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Phys.494 Applied Fourier Analysis

Spring 2008

Homework II

Complex integration

1. a) Obtain the series expansion of

$$f(z) = \frac{sinhz}{z^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n+1)!} \; .$$

b) Show that the contour integral, $\oint dz\, f(z) = 2\pi i$, if the contour encloses the origin.

2. One way to define the Hermite polynomials is by their contour integral representation:

$$H_n(z) = \frac{2^n n!}{2\pi i} \oint dt\, e^{zt} \frac{e^{-t^2/4}}{t^{n+1}} \tag{1}$$

where $n = 0, 1, 2, \ldots$ and the contour encircles the origin.

a) Compute $H_n(z)$ for $n = 0, 1, 2$.

b) Compute the function

$$G(z,h) \equiv \sum_{n=0}^{\infty} H_n(z) h^n / n! \tag{2}$$

by interchanging summation and the contour integration and obtain the generating function of Hermite polynomials.

$$G(z,h) = e^{2zh-h^2}. \tag{3}$$

3. One way to define the Laguerre polynomials is by their contour integral representation:

$$L_n(z) = \frac{1}{2\pi i} \oint dt\, e^{zt} \frac{(t-1)^n}{t^{n+1}} \tag{4}$$

where $n = 0, 1, 2, \ldots$ and the contour encircles the origin.

a) Compute $L_n(z)$ for $n = 0,\, 1,\, 2$.

b) Compute the function

$$G(z,h) \equiv \sum_{n=0}^{\infty} h^n L_n(z) \tag{5}$$

by interchanging summation and the contour integration and obtain the generating function of Laguerre polynomials.

$$G(z,h) = \frac{1}{1-h} e^{-\frac{zh}{1-h}}. \tag{6}$$

4. Consider the famous integral:

$$I = \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, e^{-\frac{x^2}{2\sigma^2}}\,. \tag{7}$$

Consider the contour $-R, +R, R - i\Lambda, -R - i\Lambda$. Complete the squares in the exponent. You must decide on Λ to reduce integral to a simple Gaussian integral. Obtain the result:

$$I = \sqrt{2\pi}\,\sigma\, e^{-\frac{k^2\sigma^2}{2}}\;\;. \tag{8}$$

5. Take the integral:

$$\int_0^{\pi} d\theta\, \frac{1}{a + bcos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad \text{if } \; a > b > 0\,.$$

Take the contour as the unit circle and let $z = e^{i\theta}$.

6. Take the integral:

$$\int_0^{\infty} dx\, \frac{1}{1+x^4} = \frac{\pi\sqrt{2}}{4}\,.$$

7. Take the integral:

$$\int_0^{\infty} dx\, \frac{sinx}{x} = \frac{\pi}{2}\,.$$

Hint: Compute $\oint dz\, \frac{e^{iz}}{z}$ by indenting the contour around $z = 0$ to avoid the pole there, by a small circle of radius a, and let a go to zero after taking the integral.