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Phys.494 Applied Fourier Analysis

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Homework I

Complex numbers and functions

1. If $z_1 = a + ib$ and $z_2 = c + id$ show that

$$
\begin{aligned}
z_1 + z_2 &= a + c + i(b + d) \\
z_1\, z_2 &= ac - bd + i(ad + bc) \\
|z_1| &= \sqrt{z_1 z_1^*} = \sqrt{a^2 + b^2} \\
|z_2| &= \sqrt{c^2 + d^2} \\
\frac{z_1}{z_2} &= \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \\
z &= \frac{(-1 + 3i)(1 + 2i)}{2 - i} + 2i = -3 + i
\end{aligned}
$$

2. We proved in class that $e^{i\theta} = cos\theta + isin\theta$. Show that

$$
\begin{aligned}
cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
cos\, i\theta &= cosh\,\theta \\
sin\, i\theta &= isinh\,\theta \\
(cos\theta + isin\theta)^n &= cos\, n\theta + i\, sin\, n\theta \\
cos\, n\theta &= cos^n\,\theta - \frac{n(n-1)}{2} cos^{n-2}\,\theta\; sin^2\,\theta + \ldots
\end{aligned}
$$

(hint: for the last equality expand $(cos\theta + isin\theta)^n$ using the binomial theorem.

3. The following problem presents an important mathematical operation that is very useful in the physics of vibrations and waves, namely in the *beat* and *interference* phenomena.

a) Show that

$$e^{i\theta_1} + e^{i\theta_2} = \left\{ 2\; cos\left(\frac{\theta_2 - \theta_1}{2}\right)\right\}\; e^{i\frac{(\theta_1+\theta_2)}{2}}\;.$$

Note that for $\theta_2 - \theta_1 = 2n\pi$, n integer, the amplitude in curly brackets has magnitude 2, and for $\theta_2 - \theta_1 = (2n+1)\pi$ the amplitude is 0.

b) For the case of complex amplitudes a and b,

$$a\, e^{i\theta_1} + b\, e^{i\theta_2} = z$$

show that maximum and minimum possible values of z are: $|z|_{max} = |a| + |b|$ and $|z|_{min} = |(|a| - |b|)|$.

4.a Show the following complex numbers as vectors in the complex z-plane. Find also their magnitudes and phases.

$$\begin{aligned} z_1 &= 3 + 2i \quad \text{ans. } \rho = \sqrt{13}\,, \quad \theta = 33.7 \text{ deg.} \\ z_2 &= 2 - 3i \\ z_3 &= 3e^{i\pi/4} \end{aligned}$$

4.b Interpret geometrically the result of multiplying one complex number $Ae^{i\theta}$ by another complex number $Be^{i\phi}$. Show all three numbers as vectors in the z-plane.

5.A Let $z(t) = Ae^{i\omega t}$ with $A = 5$
a) Plot z as a function of time (draw the vector in the z-plane), for t values:

$$t = 0, \quad \frac{1}{4}\frac{2\pi}{\omega}, \quad \frac{1}{2}\frac{2\pi}{\omega}, \quad \frac{3}{4}\frac{2\pi}{\omega}, \quad 1\,\frac{2\pi}{\omega}.$$

b) Plot $x = Re\, z$ as a function of time.
c) Plot $y = Im\, z$ as a function of time.
5.B Repeat for $z(t) = Ae^{i\omega t}$ with $A = 5e^{i\pi/3}$
5.C Repeat for $z(t) = 5e^{-i\omega t}$
5.D Repeat for $z(t) = 5e^{i\pi/3}e^{-i\omega t}$.

6. Inverse trigonometric functions

$$z = sin\phi \qquad \phi = sin^{-1}z = -i\, ln(iz + \sqrt{1-z^2})$$
$$z = cos\phi \qquad \phi = cos^{-1}z = -i\, ln(z + \sqrt{z^2-1})$$
$$z = tan\phi \qquad \phi = tan^{-1}z = \frac{i}{2}ln\left(\frac{i+z}{i-z}\right)$$

7. Inverse hyperbolic functions

$$z = sinh\phi \qquad \phi = sinh^{-1}z = ln(z + \sqrt{1+z^2})$$
$$z = cosh\phi \qquad \phi = cosh^{-1}z = ln(z + \sqrt{z^2-1})$$
$$z = tanh\phi \qquad \phi = tanh^{-1}z = \frac{1}{2}ln\left(\frac{1+z}{1-z}\right)$$

8. Time Averages Let $A(t) = A_o e^{i\omega t}$ and $B = B_o e^{i\omega t}$ where A_o and B_o are constant complex amplitudes. The physical quantities are real, $A_{phys}(t) = Re.A(t)$, and similarly for B. Define the time average of a function of time $f(t)$ as

$$< f(t) >= \frac{1}{T} \int_0^T dt\, f(t)$$

where $T = 2\pi/\omega$ is the period of oscillation. Show that

$$\begin{aligned}
< A_{phys}(t) > &= \; = 0 \\
< B_{phys}(t) > &= \; = 0 \\
< A_{phys}(t)\, B_{phys}(t) > &= \frac{1}{2} Re.A_o\, B_o^* = \frac{1}{2} Re.A_o^*\, B_o \\
< A_{phys}^2(t) > &= \frac{1}{2} |A_o|^2.
\end{aligned}$$