MUSICAL FOURIER

Physics 494 Applied Fourier Analysis

Spring 2008

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Chapter 1

Fourier Series

1.1 Basis Vectors

Consider the eigenvalue equation represented by the differential equation,

$$L(q)y_\lambda(q) = \lambda y_\lambda(q), \tag{1.1}$$

where $L(q) = -D^2$, $D = \frac{d}{dq}$, λ is an eigenvalue and $y_\lambda(q)$ is an eigenfunction of the operator $L(q)$ with eigenvalue λ. We are searching for solutions that satisfy the *periodic* boundary conditions:

$$y_\lambda(a) = y_\lambda(b) \tag{1.2}$$

$$y'_\lambda(a) = Dy_\lambda(q)|_a = y'_\lambda(b) \tag{1.3}$$

a) Let us first consider the formal properties of this system (DE and BC) [1]: We note, after an integration by parts, that

$$\int_a^b dq\, y_\mu^* L(q) y_\lambda = y_\mu^* \left(-Dy_\lambda\right)|_a^b + \int_a^b dq \, \left(Dy_\mu^*\right) \left(Dy_\lambda\right), \tag{1.4}$$

and the surface term (first term on the right-hand-side of (1.4)) vanishes because of the periodic boundary conditions (1.2) and (1.3). Thus,

$$\begin{aligned} \int_a^b dq \, \left\{y_\mu^* L(q) y_\lambda - y_\lambda L(q) y_\mu^*\right\} &= 0, \\ &= (\lambda - \mu^*) \int_a^b dq\, y_\mu^*(q)\, y_\lambda(q)\,. \end{aligned} \tag{1.5}$$

Consider the case $\mu = \lambda$. Eq.(1.5) reads now

$$(\lambda - \lambda^*) \int_a^b dq \, |y_\lambda|^2 = 0. \tag{1.6}$$

Since the integral for non-vanishing solutions is positive, we conclude that the eigenvalues must be real, i.e., $\lambda = \lambda^*$. Next consider the case of $\mu \neq \lambda$, Eq.(1.5) reads

$$(\lambda - \mu) \int_a^b dq \, y_\mu^* \, y_\lambda = 0, \tag{1.7}$$

which says that the eigenfunctions that belong to different eigenvalues are orthogonal. If on the other hand $\mu = \lambda$, then we can set the magnitude of the integral, arbitrarily, equal to unity:

$$\int_a^b dq \, |y_\lambda|^2 = 1. \tag{1.8}$$

What if there exists degeneracy, solutions with the same value of an eigenvalue? In this case, provided that the degenerate eigenfunctions are linearly independent, and if they are not already orthogonal, we can orthonormalize them, say via Gram-Schmidt orthonormalization procedure. In our case, as you will see below, $\lambda < 0$ is not an eigenvalue, $\lambda = 0$ is an eigenvalue and is non-degenerate, and $\lambda > 0$ is a doubly-degenerate eigenvalue. Now let us discuss the eigenvalues and eigenfuctions [2] for this system (DE and BC) :

1. Show that for $\lambda < 0$ the solution

$$y_\lambda(q) = c_1 e^{\sqrt{-\lambda} q} + c_2 e^{-\sqrt{-\lambda} q} \tag{1.9}$$

   satisfies the differential equation (1.1) , but not the periodic boundary conditions (1.2) and (1.3), hence is discarded.

2. Show that for $\lambda = 0$ the solution ,

$$y_\lambda(q) = c_1 + c_2 \, q \tag{1.10}$$

   satisfies the differential equation *and* the periodic boundary conditions for c_1 an arbitrary constant and $c_2 = 0$.

3. Show that for $\lambda > 0$ the solution ,

$$y_\lambda(q) = c_1 cos\sqrt{\lambda}\, q + c_2 sin\sqrt{\lambda}\, q \tag{1.11}$$

satisfies the differential equation *and* the periodic boundary conditions but only for certain values of the parameter λ: If we write the periodic boundary conditions with the solution (1.11) , then for the homogeneous set of equations for the coefficients c_1, c_2 we obtain $M\mathbf{c} = 0$ where :

$$M = \begin{bmatrix} cos\sqrt{\lambda}a - cos\sqrt{\lambda}b & sin\sqrt{\lambda}a - sin\sqrt{\lambda}b \\ -\sqrt{\lambda}(sin\sqrt{\lambda}a - sin\sqrt{\lambda}b) & \sqrt{\lambda}(cos\sqrt{\lambda}a - cos\sqrt{\lambda}b) \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{1.12}$$

For nonvanishing $\mathbf{c}$ the determinant of the matrix M has to be zero:

$$detM = 4\sqrt{\lambda}\, sin^2 \left\{ \sqrt{\lambda} \left(\frac{b-a}{2} \right) \right\} = 0. \tag{1.13}$$

Equation 1.13 determine the eigenvalues λ. Excluding the case of $\lambda = 0$ which was studied above seperately, we obtain,

$$\sqrt{\lambda_n} = \frac{2n\pi}{L} \text{ where } n = 1, 2, \ldots, \infty \text{ , and } L \equiv b - a \ . \tag{1.14}$$

Thus $\lambda_o = 0$ is non-degenerate (only one eigenfunction belonging to that eigenvalue) and all other eigenvalues, $\lambda_{n>0}$, are doubly degenerate (two eigenfunctions (sinus and cosinus functions) belonging to that eigenvalue).
We shall consider $y_\lambda(q)$ as the q^{th} component of the vector $|y_\lambda>$ and represent it as $<q|y_\lambda>$.
We now represent the eigenvectors of the operator $L(q) = -\frac{d^2}{dq^2}$ by:

$$\begin{aligned} <q|1_o> &= \frac{1}{\sqrt{L}} \\ <q|C_n> &= \sqrt{\frac{2}{L}}\, cos\frac{2n\pi q}{L} \\ <q|S_n> &= \sqrt{\frac{2}{L}}\, sin\frac{2n\pi q}{L} \end{aligned} \tag{1.15}$$

Now define the scalar product (overlap) of two vectors $|f>$ and $|g>$ in the interval $a \leq q \leq b$ as follows

$$< f|g >= \int_a^b f^*(q)\, g(q)\, dq \; =< g|f >^* \;\; . \tag{1.16}$$

where * denotes complex conjugation. Show that eigenvectors are an orthonormal set (take the overlap integrals explicitly)

$$\begin{aligned}
< 1_o|1_o > &= 1 \\
< C_n|C_n > &= 1 \\
< S_n|S_n > &= 1 \\
< 1_o|S_n > &= 0 \\
< 1_o|C_n > &= 0 \\
< S_n|C_m > &= 0 \\
< S_n|S_m > &= \delta_{n\,m} \\
< C_n|C_m > &= \delta_{n\,m}
\end{aligned} \tag{1.17}$$

b) An arbitrary vector $|f >$ (satisfying Dirichlet's conditions to be listed later) can be expanded as:

$$|f >= a_o|1_o > + \sum_{n=1}^{\infty} (a_n|C_n > + b_n|S_n >) \tag{1.18}$$

Show that the expansion coefficients are:

$$\begin{aligned}
a_o &= < 1_o|f >= \int_a^b dq \sqrt{\frac{1}{L}}\, f(q) \\
&= \sqrt{\frac{1}{L}} \text{ area under } f(q) \\
a_n &= < C_n|f >= \int_a^b dq \sqrt{\frac{2}{L}} cos \frac{2n\pi q}{L}\, f(q) \\
b_n &= < S_n|f >= \int_a^b dq \sqrt{\frac{2}{L}} sin \frac{2n\pi q}{L}\, f(q)
\end{aligned} \tag{1.19}$$

Insert the coefficients a_o, a_n and b_n in place: So Eq. (1.18) looks like this:

$$\begin{aligned}
|f > &= 1_{op}|f > \quad \text{where the unit operator } 1_{op} \text{ is given as,} \\
1_{op} &= |1_o >< 1_o| + \sum_{n=1}^{\infty} (|C_n >< C_n| + |S_n >< S_n|)
\end{aligned} \tag{1.20}$$

(Note that $< C_n|C_m >$ is a number, but $|C_n >< C_m|$ is an operator. An operator acts on a vector and transforms it into another vector).
Show that

$$1_{op}.1_{op} = 1_{op} \tag{1.21}$$

And also note that:

$$< q|f >=< q|1_{op}|f >= \int_a^b < q|1_{op}|q' >< q'|f > dq' \tag{1.22}$$

Thus

$$\begin{aligned} < q|1_{op}|q' > &= < q|q' >= \delta(q-q') \quad \text{such that} \\ \int_a^b dq'\ \delta(q-q') &= 1 \quad \text{where } q \text{ and } q' \text{ are in the interval } [a,b]\ . \end{aligned} \tag{1.23}$$

We shall study the properties of the Dirac delta function $\delta(q - q')$ later.

With the overlap or scalar product of two vectors so defined we can rewrite the Eqs.(1.5) and complete the survey of the formal properties of this system:

$$< y_\mu|(\mathcal{L}y_\lambda) >=< (\mathcal{L}y_\mu)|y_\lambda > . \tag{1.24}$$

where the matrix elements of the local operator $\mathcal{L}$ are given by:

$$< q|\mathcal{L} = L(q) < q| . \tag{1.25}$$

Any matrix A satisfies the following relation (show this):

$$< \psi|A\phi >=< A^\dagger\psi|\phi >, \tag{1.26}$$

where $A^\dagger$ is Hermitian conjugate of A, defined as $(A^\dagger)_{ij} \equiv A^*_{ji}$. Hence under periodic boundary conditions our differential operator $L(q)$ is Hermitian. Or we may just say $\mathcal{L} = \mathcal{L}^\dagger$. Let us write down once again our equations in this format:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^\dagger \\ < q'|\mathcal{L}|q > &= L(q')\delta(q'-q) = -\frac{d^2}{dq'^2}\delta(q'-q) \\ \mathcal{L}|y_\lambda > &= \lambda|y_\lambda > \quad \text{with } \lambda = \lambda^* \\ < y_\mu|y_\lambda > &= \delta_{\mu\lambda} \quad \text{eigenvectors are orthonormal} \\ 1_{op} &= \sum_\lambda |y_\lambda >< y_\lambda| \quad \text{eigenvectors are complete.} \end{aligned} \tag{1.27}$$

The possibility of expanding arbitrary vectors in terms of the eigenvectors of the Hermitian operator $\mathcal{L}$ results in a great number of important applications. If the function $f(q)$ is periodic, the $n = 1$ terms in the expansion Eq. (1.18) and higher order terms denote a *fundamental* property and *overtones* respectively. If the q axis denotes *distance*, we rename it as x, if it denotes *time* we rename it as t . In the first case the period $b - a = L$ is the length of the fundamental wavelength λ. Higher order terms $(n > 1)$ have wavelengths $\lambda_n = \lambda/n$. And we may define a wavenumber $k_n = 2\pi/\lambda_n$. Thus $< x|C_n >\propto cos k_n x$. In the second case $b-a = T$ is the fundamental period ($f = 1/T$ is the fundamental rotational frequency.). Higher order terms $(n > 1)$ have periods $T_n = T/n$ (or frequency $f_n = nf$) . We may define an angular frequency $\omega_n = 2\pi/T_n = 2\pi fn = \omega_o n$. Thus $< t|C_n >\propto cos\, n\omega_o t$ so that the n th mode has n times the fundamental rotational frequency (or angular frequency). Also note that one may use as variable $\theta = 2\pi x/L$ or $\theta = 2\pi t/T$ in radians , in this case $[-\pi, +\pi]$ is usually taken as the boundary positions of the periodic cell, because if there is symmetry in the problem it is handled easier. Same with x boundaries $[-\frac{L}{2}, +\frac{L}{2}]$ and t boundaries $[-\frac{T}{2}, +\frac{T}{2}]$

Finally, for the sake of completeness let us list the Dirichlet's conditions. A periodic function $f(q)$ of period L satisfying the following conditions has a convergent Fourier series. These conditions are:

1. f(q) is absolutely integrable:

$$\int_q^{q+L} dq' \, |f(q')| < \infty \quad ,$$

2. f(q) has only finite number of maxima and minima in any range of length L,

3. The number of discontinuities must be finite, in any range of length L.

If $f(q)$ satisfies the Diriclet's conditions its Fourier series converges. The sum of the series is $f(q)$, except at any point q_o at which $f(q)$ is discontinuous. At a point of discontinuity the function assumes the value:

$$f(q_o) = \frac{1}{2} \left[f(q_o + 0) + f(q_o - 0) \right] .$$

Now some practical problems:

1) Let the periodic cell's boundaries be $A = -L/2$ and $B = +L/2$. A rectangular pulse $|f >$ of height H extends from $x = -a/2$ to $x = +a/2$ with $a < L$.
i)Expand it in Fourier series and compute the first two non-vanishing expansion coefficients.
Answer:

$$< x|f > \approx \frac{aH}{\sqrt{L}} < x|1_o > + \frac{H\sqrt{2L}}{\pi} sin\frac{\pi a}{L} < x|C_1 > + \dots \text{ or,}$$

$$< x|f > \approx \frac{aH}{L} + \frac{2H}{\pi} sin\frac{\pi a}{L} cos\frac{2\pi x}{L} \ .$$

ii) the n^{th} order term $A_n \ cos\frac{2n\pi x}{L}$
Answer:

$$A_n = 2H\frac{sin\frac{n\pi a}{L}}{n\pi}$$

iii) Write the expansion for the square pulse ($a = L/2$).
Answer:

$$f(x) = H\left[\frac{1}{2} + \frac{2}{\pi}\left(cos\frac{2\pi x}{L} - \frac{1}{3}cos3\frac{2\pi x}{L} + \frac{1}{5}cos5\frac{2\pi x}{L} + \dots\right)\right]$$

iv) Consider the result of part (ii). What is the general expression for A_n/A_1?
Answer:

$$\frac{A_n}{A_1} = \frac{sin\, n\pi\lambda}{n sin\, \pi\lambda},$$

where $\lambda \equiv \frac{a}{L}$ is a measure of the width of the pulse as compared to the width of the periodic cell. The figures (1.1) and (1.2) plot this ratio as a function of λ for $n = 13$ and $n = 12$ respectively. Show that this ratio is symmetric about $\lambda = 1/2$ for $n = n_{odd}$ and antisymmetric about $\lambda = 1/2$ for $n = n_{even}$.

Repeat plotting these figures for $n = 101$ and $n = 100$. Try a few other values as well. Thus the ratio in absoulte magnitude approaches unity for $\lambda \to 0^+$, or 1^-, which can also be shown analytically in addition to the figures. These limits mean that as the pulse width aprroaches zero, or approaches the the width of the periodic cell. The ratio A_n/A_1 gets smaller for $a \approx L/2$. The conclusion is that as the signal gets *spiky*, convergence becomes poor. Alp Sipahigil rightly commented in class [3] that the limit of a very *wide* rectangular pulse train behaves in just the same way like a very *sharp* pulse train.

v) BE CAREFUL: Differentiate the Fourier series of the square pulse term

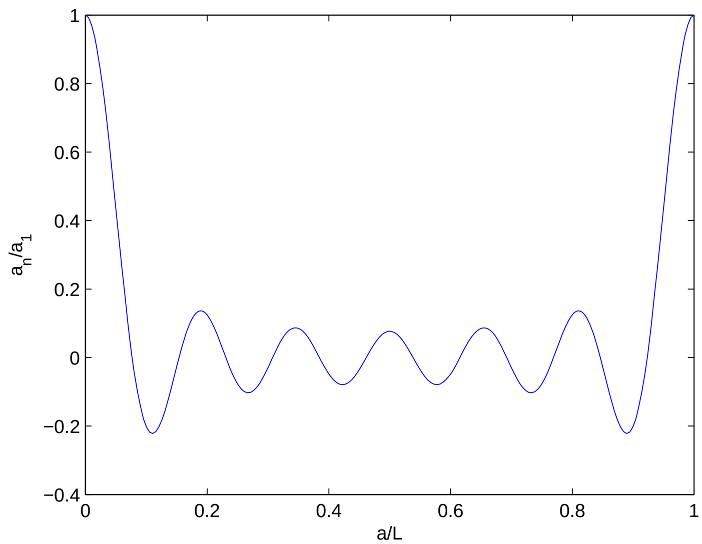


Figure 1.1: n=13

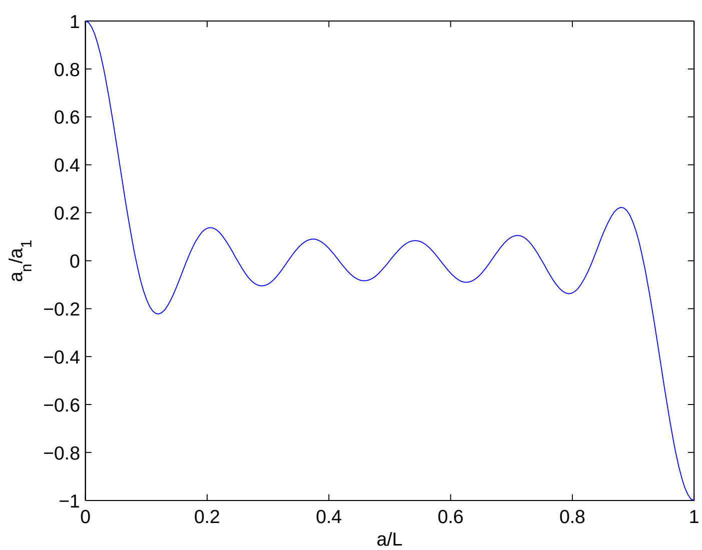


Figure 1.2: n=12

by term. Does the resulting series converge or not?
Hint:
Show that a general term $a_{2n+1}\frac{d}{dx} < x|C_{2n+1} >$ does not vanish as n approaches infinity, a necessary but not sufficient condition for the convergence of infinite series !! However, you can integrate Fourier series term by term and the resulting series converges even more rapidly!! Hence, be careful, in general, when you differentiate a Fourier series, it may or may not converge, you have to check the convergence!!

2) Use the fact that $< f|f >=< f|1_{op}|f >$ and so:

$$< f|f >= a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

which is known as the *Parseval's relation.*

3) Replace the rectangular pulse of Problem 1 above with an isosceles-triangular pulse.That is, let the boundaries of the periodic cell be $-L/2$ and $+L/2$. An isosceles-triangular pulse $|f >$ of height H has a base extending from $q = -a/2$ to $q = +a/2$ with $a < L$.
Repeat the questions of the rectangular pulse train for this triangular pulse.
i) show that the Fourier coefficients (Eq.(1.19) are,

$$\begin{aligned} a_o &= \frac{Ha}{2\sqrt{L}} \text{ and,} \\ a_n &= \sqrt{\frac{2}{L}}\left(\frac{Ha}{2}\right)\left(\frac{sin\frac{n\theta_o}{2}}{\frac{n\theta_o}{2}}\right)^2 \quad \text{where } \theta_o = \tfrac{\pi a}{L} \\ b_n &= 0\,. \end{aligned}$$

(You need to consider the integrals only from $q = 0$ to $q = a/2$ when you compute the expansion coefficients, why?)
ii) Compute the ratio for $a_{n=101}/a_1$ for this triangular pulse numerically:
First take the pulse width $\theta_o = \pi$ radians , $(a = L)$,
Answer:

$$\frac{a_{n=101}}{a_1} \approx 10^{-4}.$$

Next take a narrow pulse width, $2\theta_o = 0.02$ radians,$(a = L/100\pi)$,
Answer:

$$\frac{a_{n=101}}{a_1} \approx 0.92 \ \ .$$

It is important again to compare the two numerical values. If the pulse width is large, convergence is rapid and a few terms (say 10) would be enough for the representation of the pulse. If the pulse width is very narrow, convergence is very slow and one would have to consider a very large number of terms.

1.2 Change of Basis

A new set of basis vectors can be constructed by taking linear combinations of the old ones :

$$\begin{aligned} |Z_o > &= |1_o > \\ |Z_{n+} > &= \frac{1}{\sqrt{2}}\left(|C_n > +i|S_n >\right) \quad \text{n=1,2,...} \\ |Z_{n-} > &= \frac{1}{\sqrt{2}}\left(|C_n > -i|S_n >\right) \quad \text{n=1,2,...} \\ < q|Z_{n+} > &= \frac{1}{\sqrt{L}} e^{\frac{2\pi i n q}{L}}, \\ < q|Z_{n-} > &= \frac{1}{\sqrt{L}} e^{-\frac{2\pi i n q}{L}}. \end{aligned} \tag{1.28}$$

Show that the completeness relation Eq. (1.20) now reads as:

$$1_{op} = |Z_o >< Z_o| + \sum_{n=1}^{\infty} \left(|Z_{n+} >< Z_{n+}| + |Z_{n-} >< Z_{n-}|\right). \tag{1.29}$$

We can achieve economy of notation by relabelling the basis vectors:

$$\begin{aligned} |Z_n > &= |Z_{n+} > \\ |Z_{-n} > &= |Z_{n-} > \end{aligned} \tag{1.30}$$

Show that the equation Eq. (1.29) can be written in a compact form:

$$1_{op} = \sum_{n=-\infty}^{+\infty} |Z_n >< Z_n|. \tag{1.31}$$

In Eq. (1.31) if we change the index n to $-n$ we can write the completeness relation in the following form as well

$$1_{op} = \sum_{n=-\infty}^{+\infty} |Z_{-n} >< Z_{-n}|, \tag{1.32}$$

where we used the fact that in a summation, order of the terms summed does not matter. Infact, we can still simplify our notation, let $|Z_n >= |n >$ and $|Z_{-n} >= | - n >$ and the unity operator can be written as:

$$1_{op} = \sum_{n=-\infty}^{+\infty} |n >< n| = \sum_{n=-\infty}^{+\infty} | - n >< -n|. \tag{1.33}$$

Prove the orthonormality of this new set of basis vectors:

$$< n|m >= \delta_{n\,m}. \tag{1.34}$$

We can expand a reasonable vector (satisfying Dirichlet's conditions) in terms of this new basis vectors, result is the *complex Fourier series*:

$$\begin{aligned} |f > &= \sum_n c_n |n > \\ < q|1_{op}|f >= f(q) &= \sum c_n < q|n >= \sum c_n \frac{e^{\frac{2\pi i q n}{L}}}{\sqrt{L}} \\ &, \end{aligned} \tag{1.35}$$

$$\text{where} \quad c_n =< n|f > = \int_a^b dq' \frac{e^{-\frac{2\pi i q' n}{L}}}{\sqrt{L}} f(q') \tag{1.36}$$

Note that if $f(q)$ is a real function then, $c_{-n} = c_n^*$. The complex Fourier series is useful for formal mathematical manipulations because of the simplicity of the exponential function, however, in order to compute the expansion coefficients numerically, real form is preferred generally.

As a simple example of a complex representation of a signal, consider a time signal which is a pure sinusoid, a wave with a fixed frequency (for simplicity we now lump the $\frac{1}{\sqrt{T}}$ factor into the modulus of the complex constant c):

$$f(t) = ce^{i\omega t} + c^* e^{-i\omega t}. \tag{1.37}$$

Exercise: what is c in the above equation if

$$\begin{aligned} i) \quad f(t) &= |A|cos(\omega t + \phi). \\ ii) \quad f(t) &= |A|sin(\omega t + \phi). \end{aligned} \tag{1.38}$$

We can also represent a sinusoid as $f(t) = Re\hat{f} = Re\{ce^{i\omega t}\}$.
Again,find c in the above equation if

$$\begin{aligned} i) \quad f(t) &= |A|cos(\omega t + \phi). \\ ii) \quad f(t) &= |A|sin(\omega t + \phi). \end{aligned} \tag{1.39}$$

Problem set: 1

1. Function $f(x)$ is given as:

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0\,; \\ \;\;k & \text{if} \quad 0 < x < \pi\,, \end{cases} \tag{1.40}$$

where $f(x) = f(x + 2\pi)$. Expand this function in Fourier series.
Ans.:

$$f(x) = \frac{4k}{\pi}\left(sinx + \frac{1}{3}sin3x + \frac{1}{5}sin5x + \ldots\right)\,.$$

Write a script m-file to add two, three,...,ten terms numerically, and show the graph for $0 < x < 7\pi$ on a Matlab plot.
For $k = 1$ and $x = \frac{\pi}{2}$ obtain the interesting sum:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}\,.$$

2. Function $f(x)$ is given as:

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x < -\frac{\pi}{2}\,; \\ 1 & \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}\,; \\ 0 & \text{if} \quad \frac{\pi}{2} < x < \pi\,, \end{cases} \tag{1.41}$$

where $f(x) = f(x + 2\pi)$. Expand this function in Fourier series.
Ans.:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(cosx - \frac{1}{3}cos3x + \frac{1}{5}cos5x + \ldots\right)\,.$$

Write a script m-file to add two, three,..., ten terms numerically, and show the graph for $0 < x < 7\pi$ on a Matlab plot.

3. Function $f(x)$ is given as:

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1\,; \\ k & \text{if} \quad -1 < x < 1\,; \\ 0 & \text{if} \quad 1 < x < 2\,, \end{cases} \tag{1.42}$$

where $f(x) = f(x + 4)$. Expand this function in Fourier series.
Ans.:

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(cos\frac{\pi x}{2} - \frac{1}{3}cos\frac{3\pi x}{2} + \frac{1}{5}cos\frac{5\pi x}{2} + \ldots\right)\,.$$

Write a script m-file to add two, three,...,ten terms numerically, and show the graph for $0 < x < 10$ on a Matlab plot.

4. Square wave Function $f(t)$ is given as:

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 < t < \frac{T}{2}\,; \\ -1 & \text{if} \quad \frac{T}{2} < t < T\,, \end{cases} \tag{1.43}$$

where $f(t) = f(t+T)$. a) Expand this function in Fourier series.
Ans.:

$$f(t) = \sum_{n=1,3,5...}^{\infty} \frac{4}{\pi n} sin(n\omega_o t)\ , \quad \omega_o = \frac{2\pi}{T}\ .$$

Take $T = 225$ sec. Write a script m-file to add nine harmonics, then again upto 39^{th} harmonic numerically, and show the graph for $0 < t < 500$ secs. on a Matlab plot. Comment upon the Gibbs oscillations.
b) Compute the spectrum of a square wave, the magnitude of the $|b_n|$ coefficients, upto harmonic number $n = 50$, and make a Matlab plot.

5. Sawtooth wave Function $f(x)$ is given as:

$$f(x) = \Big\{\ x + \pi \quad \text{if} \quad -\pi < x < \pi\,, \tag{1.44}$$

where $f(x) = f(x+2\pi)$. Expand this function in Fourier series.
Ans.:

$$f(x) = \pi + 2\left(sinx - \frac{1}{2}sin2x + \frac{1}{3}sin3x + \dots\right)\ .$$

Write a script m-file to add two, three,...,ten terms numerically, and show the graph for $0 < x < 7\pi$ on a Matlab plot.

6. Triangular wave Function $f(t)$ is given as:

$$f(t) = \begin{cases} t - \frac{T}{4} & \text{if} \quad 0 < t < \frac{T}{2}\,; \\ \\ -t + \frac{3T}{4} & \text{if} \quad \frac{T}{2} < t < T\,, \end{cases} \tag{1.45}$$

where $f(t) = f(t+T)$.
a) Expand this function in Fourier series.
Ans.:

$$f(t) = -\sum_{n=1,3,5...}^{\infty} \frac{2T}{\pi^2 n^2} cos(n\omega_o t)\,, \quad \omega_o = \frac{2\pi}{T}\ .$$

b) Take $T = 225$ sec. Write a script m-file to add two, three,...,ten terms numerically, and show the graph for $0 < t < 500$ secs. on a Matlab plot.
c) Compute the spectrum of a triangular wave, the magnitude of the $|a_n|$ coefficients, upto harmonic number $n = 50$, and make a Matlab plot. Compare your result with that of problem 4.c.

7. Periodic impulse Function $f(t)$ is given as:

$$f(t) = \begin{cases} \frac{2}{\epsilon} & \text{if} \quad 0 < t < \frac{\epsilon}{2}\,; \\ 0 & \text{if} \quad \frac{\epsilon}{2} < t < \frac{T}{2} - \frac{\epsilon}{2}\,; \\ -\frac{2}{\epsilon} & \text{if} \quad \frac{T}{2} - \frac{\epsilon}{2} < t < \frac{T}{2} + \frac{\epsilon}{2}\,; \\ 0 & \text{if} \quad \frac{T}{2} + \frac{\epsilon}{2} < t < T - \frac{\epsilon}{2}\,; \\ \frac{2}{\epsilon} & \text{if} \quad T - \frac{\epsilon}{2} < t < T\,, \end{cases} \tag{1.46}$$

where $f(t) = f(t + T)$. Expand this function in Fourier series and take the limit $\epsilon \to 0^+$.
Ans.:

$$f(t) = \frac{8}{T} \sum_{n=1,3,5...}^{\infty} cos(n\omega_o t)\,, \quad \omega_o = \frac{2\pi}{T}\,.$$

Discuss the n dependence of the $|a_n|$ coefficients and hence the spectrum of a periodic impulse train.

1.3 Aliasing

Consider a sinusoid $\hat{f}(t) = ce^{i\omega_o t}$ with frequency $f_o = \omega_o/2\pi$. We sample it at times $t = nT_s$ for $n = 0, \pm 1, \pm 2, \ldots$ so that we have samples of the analog (continuous) signal , $\hat{f}(n) = ce^{i\omega_o nT_s}$. Now consider another sinusoid with frequency $f_o + kf_s$, namely, $\hat{g}(t) = ce^{i2\pi(f_o+kf_s)t}$, k being an integer. If this new and different signal is sampled again with the same sampling frequency, we find $\hat{g}(n) = ce^{i2\pi(f_o+kf_s)nT_s}$. But since $f_sT_s = 1$ and $e^{2\pi ikn} = +1$, we observe that $\hat{f}(n) = \hat{g}(n)$. They are identical ! Two analog sinusoids of *different* frequencies, after sampling, are indistinguishable.
We want to find the answer to the following two questions: What is the minimum frequency contained in the signal after sampling? And what are

all the possible frequencies? Let the original frequency be f_o, such that

$$f_o = m f_s + f_r \ , \tag{1.47}$$

where m is a positive integer or zero and $0 < f_r < f_s$. Show that

$$\hat{f}(n) = c\, e^{2\pi i (m f_s + f_r) n T_s} = c\, e^{2\pi i a n} \ \text{ where } a \equiv f_r/f_s \ . \tag{1.48}$$

Let us multiply our signal $\hat{f}(n)$ as written in Eq.(1.48) by $(+1)$,

$$\hat{f}(n) = c\, e^{2\pi i a n} \times (+1) = c\, e^{2\pi i a n}\, e^{2\pi i \ell n} = c\, e^{2\pi i (a+\ell) n} \ \text{ where } \ell = 0, \pm 1, \pm 2, \ldots \tag{1.49}$$

Obviously, the minimum frequency (written in units of f_s) is either a or $a-1$, whichever is smaller in absolute value (also note that a physical frequency has to be positive (see the exercises from Eq.(1.37) to Eq.(1.39)), and since $a < 1$ we have two frequencies f_r and $f_s - f_r$ to compare). So in the complex representation, where negative frequencies are allowed, possible frequencies are: $\ldots, a-2, a-1, a, a+1, a+2, \ldots$. But in the real world we have positive frequencies only, and we get two seperate families: $a, a+1, a+2, \ldots$ and $1-a, 1-a+1, 1-a+2, \ldots$. If $0 \le a \le \frac{1}{2}$ then $f_{min} = a f_s = f_r$. But if $\frac{1}{2} < a < 1$ then $f_{min} = f_s - f_r$. Thus, written out in detail,

$$F_1 \equiv f_r, \tag{1.50}$$
$$F_2 \equiv f_s - f_r, \tag{1.51}$$

and the possible frequencies are:

$$\begin{array}{c} F_1, F_1 + f_s, F_1 + 2f_s, \ldots \\ F_2, F_2 + f_s, F_2 + 2f_s, \ldots \end{array} \tag{1.52}$$

And the minimum frequency resulting from sampling is:

$$f_{min} = \text{Min}\,[f_r, f_s - f_r]. \tag{1.53}$$

Eq.(1.53) explains the importance of the *Nyquist frequency* $f_N = f_s/2$, and why it is also called the *folding frequency*. If f_r defined in Eq.(1.47) is greater than the Nyquist frequency (i.e., $f_r = f_N + \delta$, $\delta > 0$), then we can write

$$\begin{aligned} f_{min} = f_s - f_r &= 2f_N - f_r \\ &= f_N - (f_r - f_N) = f_N - \delta \ . \end{aligned} \tag{1.54}$$

Thus the lowest frequency is obtained as the reflection of f_r about the folding frequency f_N, which is the Nyquist frequency defined as half the sampling rate f_s. Few simple exercises will illustrate the point. Find the lowest frequency in the following sinusoids if they are sampled with frequency f_s and also write the frequency families based on F_1 and F_2 above,Eq.(1.50):

i) $f(t = nT_s) = cos(2\pi f_o nT_s) = cos(2\pi \frac{f_o}{f_s} n) = cos(\frac{7\pi}{2} n)$.
Answer:

$$cos(\frac{7\pi}{2} n) = cos(2\pi \frac{7}{4} n),$$
$$\frac{7}{4} = 1 + \frac{3}{4}$$
$$f_r = \frac{3}{4} f_s$$
$$f_s - f_r = \frac{1}{4} f_s$$
$$\text{Min}\,[f_r, f_s - f_r] = f_s - f_r = \frac{1}{4} f_s$$

Hence the original analog signal is aliased into the minimum frequency function as a result of uniform sampling . The two frequency families contained in the sampled signal are:

$$\frac{1}{4} f_s,\ \frac{5}{4} f_s,\ \frac{9}{4} f_s,\ \ldots \quad \text{and,}$$
$$\frac{3}{4} f_s,\ \frac{7}{4} f_s,\ \frac{11}{4} f_s,\ \ldots\ .$$

Also note how the frequency folds over: the frequency $f_r = f_N + \delta$, where $f_N = f_s/2$ is the Nyquist frequency, and $\delta = f_s/4$. The frequency folds down and $f_{min} = f_N - \delta = f_s/4$.

ii) $f(t = nT_s) = cos(2\pi f_o nT_s) = cos(\frac{2\pi}{3} n)$ where $n = 0, \pm 1, \pm 2, \ldots$.
Answer:
Since the sampling frequency is $f_s = 3f_o$, in this case our original function with frequency f_o has the minimum frequency. The frequency families are: $(f_o,\ 4f_o,\ 7f_o,\ 10f_o,\ \ldots)$ and $(2f_o,\ 5f_o,\ 8f_o,\ 11f_o,\ \ldots)$.

We see that if $f_o < \frac{1}{2} f_s$ or, in other words if the sampling frequency is higher then twice the frequency of the test signal $f_s > 2f_o$ then the lowest frequency function is the original signal. And all the alias frequencies are higher. In other words since in this case $T_s < \frac{1}{2}T$, at least two samples are taken in each cycle of the original signal.

iii) Find the lowest aliased frequency of $f(t = nT_s) = cos(2\pi f_o nT_s + \phi) = cos(\frac{8\pi}{3}n + \frac{\pi}{3})$.

iv) Suppose we have two sinusoids, one the signal at 100 Hz, and one a disturbance at 200 Hz. We suspect (wrongly) that the highest frequency in the data is 125 Hz and decide to sample the data at 250 Hz. What would be the frequencies observed in the sampled data below 125 Hz? Which ones are the aliasing frequencies? Has the 200 Hz. sinusoid been aliased (or folded down) below 125 Hz.?

v) ANTIALIASING FILTER: Before making an analog to digital (ADC) conversion, a signal is passed through an *antialiasing filter*. This analog filter attenuates the frequencies in the input signal above $\frac{f_s}{2}$, half the sampling frequency. This filtered signal is then sampled. Explain why antialiasing filter needs to be used, in light of the results of the previous question (iv).

1.4 Gibbs Phenomenon

Gibbs phenomenon is about the limitation of representing a function near its discontinuity by series of eigenfunctions.
Consider the function:

$$< q|f >= \begin{cases} H & \text{for } 0 \leq q \leq \frac{L}{2}, \\ -H & \text{for } -\frac{L}{2} \leq q \leq 0. \end{cases} \tag{1.55}$$

Expand it in a complex Fourier series and show that expansion coefficients are:

$$c_n = \frac{2H\sqrt{L}}{i\pi n}\delta_{n,\ odd}. \tag{1.56}$$

So if the Fourier series is terminated, the approximation to $|f >$ is written as $|f_N >$,

$$|f >\approx |f_{N_{odd}} >= \sum_{n=-N_{odd}}^{n=N_{odd}} c_n|n > . \tag{1.57}$$

We try to compute the n-sum. Notice the following identity:

$$\frac{e^{in\theta}}{in} = \frac{1}{in} + \int_0^{\theta} d\theta' e^{in\theta'}, \tag{1.58}$$

and note that:

$$\sum_{n=-N_{odd}}^{n=N_{odd}} \frac{1}{n} = 0. \tag{1.59}$$

We now try to compute $< q|f_{N_{odd}} >$,

$$\begin{aligned} < q|f_{N_{odd}} > &= \sum_{n=-N_{odd}}^{N_{odd}} \left(\frac{2H\sqrt{L}}{i\pi n}\right) \frac{1}{\sqrt{L}} e^{i\frac{2\pi}{L}qn} \\ & \qquad \text{(define } \theta = \tfrac{2\pi q}{L} \text{ measured in radians)}, \\ &= \frac{2H}{\pi} \sum_n \left(\frac{1}{in} e^{i\theta n}\right) \\ &= \frac{2H}{\pi} \sum_n \left(\frac{1}{in} + \int_0^{\theta} d\theta' e^{in\theta'}\right), \end{aligned} \tag{1.60}$$

where in the last step we employed Eq.(1.58). The first summation vanishes due to Eq.(1.59) Now we interchange summation and the integral, and do the summation under the integral,

$$\begin{aligned} < q|f_{N_{odd}} > &= \frac{2H}{\pi} \int_0^{\theta} d\theta' e^{-iN_{odd}\theta'} \left\{1 + e^{2i\theta'} + e^{4i\theta'} + \ldots + e^{i\theta' 2N_{odd}}\right\} \\ &= \frac{2H}{\pi} \int_0^{\theta} d\theta' \frac{sin(N_{odd}+1)\theta'}{sin\theta'} = f_{N_{odd}}(\theta)\,. \end{aligned} \tag{1.61}$$

What is the first maximum of $f_{N_{odd}}(\theta)$? Just differentiate with respect to θ,

$$\frac{d}{d\theta} f_{N_{odd}}(\theta) = \frac{2H}{\pi} \frac{sin(N_{odd}+1)\theta}{sin\theta}. \tag{1.62}$$

The first maximum of $f_N(\theta)$ occurs at θ_M:

$$\theta_M = \frac{\pi}{N_{odd}+1} \quad . \tag{1.63}$$

The corresponding value for the q variable is $q_{Max} = \frac{L/2}{N_{odd}+1}$. We see that as N_{odd} increases θ_M approaches zero. In order to evalute the value of $f_N(\theta_M)$

, we make a change of variable in Eq.(1.61), $\beta = (N_{odd} + 1)\theta'$ and write the integral as follows:

$$\begin{aligned} f_N(\theta_M) &= \frac{2H}{\pi} \int_0^{\theta_M} d\theta' \frac{sin(N_{odd}+1)\theta'}{sin\theta'} \qquad (1.64) \\ &= \frac{2H}{\pi} \int_0^{\pi} d\beta \left(\frac{sin\beta}{\beta}\right) \left\{ \frac{\frac{\beta}{(N_{odd}+1))}}{sin\frac{\beta}{(N_{odd}+1)}} \right\} \quad . \end{aligned}$$

Now the integral has to be evaluated numerically. But as N_{odd} becomes very large, the term in the curly brackets in Eq.(1.64) approaches unity, $\{\ldots\} \rightarrow 1$ so that we finally have for large N_{odd},

$$f_N(\theta_M) = \frac{2H}{\pi} \int_0^{\pi} d\beta \left(\frac{sin\beta}{\beta}\right) \quad . \qquad (1.65)$$

We see that $f_N(\theta_M)$ is proportional to the *sine integral* $Si(\pi)$ where in general $Si(z)$ is defined as:

$$Si(z) = \int_0^z d\eta \left(\frac{sin\eta}{\eta}\right) \quad \text{z complex.}$$

Again this integral is to be evaluated numerically. Write a short program to evaluate the integral in Eq.(1.65), and also do it using the Matlab. Result is roughly , $\frac{2}{\pi}Si(\pi) \approx 1.18$, which means that the series overshoots the true value by about 18 per cent , but over a region of vanishingly small width of order θ_M, and after a few ripples settles down to the correct value of unity. You can find the numerical result for the Sine integral in the reference [4] or in the web-site [5]. Note that

$$\begin{aligned} f_N(\theta_M \rightarrow 0) &\approx H\, \frac{2}{\pi} Si(\pi) \approx 1.18H \\ 1.18\,H - H &= \frac{0.18}{2}\,[2H] = 0.09 \times [2H] \ , \qquad (1.66) \end{aligned}$$

where $[2H]$ represents the amount of discontinuity Eq.(1.55). Here we had a square pulse, with discontinuity $2H$ at $\theta = 0$. It can be shown, in general, that if $f(\theta)$ is smooth over the interval $[-\pi, +\pi]$ and θ_o is a point of discontinuity, then the Fourier partial sums will exhibit the same behaviour as in Eq.(1.66), with the approximate overshoot

$$0.09 \times [f(\theta_o^+) - f(\theta_o^-)] \ . \qquad (1.67)$$

Chapter 2

Fourier Integral

2.1 Definition

Now, if the boundaries of the cell move to infinity: $a \to -\infty$ and $b \to +\infty$, meaning the interval length is stretched to infinity: $L \to \infty$, then the successive $k_n = \frac{2\pi n}{L}$ values get infinitesimally close to each other. Then the summation expressing $f(q)$ as a complex Fourier series

$$< q|f >= f(q) = \sum_n < q|n >< n|f >= \sum_n c_n \frac{1}{\sqrt{L}} \, e^{\frac{2\pi i n q}{L}} \qquad (2.1)$$

can be replaced by an integral. We do this step by step now. Consider an interval of length Δn centered around the value n. The contribution of terms in that Δn band to the sum in Eq.(2.1), denoted by Δf_n is:

$$\begin{aligned} \Delta f_n &= \sum_{n \text{ in } \Delta n} c_n \frac{1}{\sqrt{L}} \, e^{\frac{2\pi i n q}{L}} \qquad (2.2) \\ &\approx \alpha \times \beta, \text{ where }, \\ \alpha &\equiv \text{ a representative value of } c_n \tfrac{1}{\sqrt{L}} \, e^{\frac{2\pi i n q}{L}} \text{ in } \Delta n \\ \beta &\equiv \text{the number of } n \text{ values in } \Delta n \end{aligned}$$

Since all integer values n are allowed (no n values are missing) the number of n values in Δn is simply $n_2 - n_1 = \Delta n$. Thus $\beta = \Delta n$. As for the α factor above, for small enough Δn and large enough L, a representative value is simply $c_n \frac{1}{\sqrt{L}} \, e^{\frac{2\pi i n q}{L}}$ because for this group of neighboring n values this factor has *almost the same value*. Thus a sum over the integers n can be transformed into a sum over the bins Δn:

$$f(q) \approx \sum_{\text{bins } \Delta n} c_n \frac{1}{\sqrt{L}} e^{\frac{2\pi i n q}{L}} \Delta n \tag{2.3}$$

At this point it is convenient to define a new variable $k \equiv \frac{2\pi n}{L}$, because as L gets increasingly larger the Δk bins get smaller. Thus we can write Eq.(2.3) as follows:

$$f(q) \approx \sum_{\text{bins } \Delta k} \frac{\Delta k}{2\pi} \left(\sqrt{L}\, c_k\right) e^{ikq} , \tag{2.4}$$

which in the limit as $L \to \infty$ can be written as an integral

$$f(q) \Longrightarrow \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left(\sqrt{L}\, c_k\right) e^{ikq} , \tag{2.5}$$

and we name the factor $\sqrt{L}\, c_k$ as the *Fourier transform* of $f(q)$ and denote it simply by $f(k)$. Its explicit representation is:

$$f(k) \equiv \sqrt{L}\, c_k = \sqrt{L} \int_{-\infty}^{+\infty} dq \frac{1}{\sqrt{L}} e^{-ikq} f(q), \tag{2.6}$$

Let us write this important formula again by making the obvious cancellations. The Fourier transform of the function $f(q)$ is :

$$f(k) = \int_{-\infty}^{+\infty} dq\, e^{-ikq} f(q), \tag{2.7}$$

Before we continue further we should pause and ask the following questions: We only let the interval L stretch to infinity,
1) What is the new form of the unity operator 1_{op} (compare Eq.(1.20)and Eq.(1.33)) ?
2) What is the new form of the orthogonality relation between the base vectors to take the place of Eq.(1.34) ?
Let us start with the question (1). We write $f(q)$ as (Convert the sum into an integral as $L \to \infty$ in just the same way we did above):

$$< q|f >= \sum_n < q|n >< n|f > \Longrightarrow < q| \left(\int_{-\infty}^{+\infty} \frac{dk}{2\pi} |k >< k| \right) |f > . \tag{2.8}$$

Thus we must have:

$$1_{op} = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} |k >< k|. \tag{2.9}$$

As for the second form of 1_{op} we compute $\sqrt{L} < n|f >$ and take the limit, we find:

$$\sqrt{L} < n|f > \Longrightarrow < k| \left(\int_{-\infty}^{+\infty} dq|q >< q| \right) |f > . \tag{2.10}$$

We must have then:

$$1_{op} = \int_{-\infty}^{+\infty} dq|q >< q|. \tag{2.11}$$

So we know now that $< q|k >= e^{ikq}$ and $< k|q >= e^{-ikq}$, but what about the scalar products (overlaps) $< q|q' >$ and $< k|k' >$? If we write the relation $1_{op}.1_{op} = 1_{op}$ first for the q-vectors, and then for the k- vectors we obtain the following relation:

$$\begin{aligned} \int_{-\infty}^{+\infty} dq' < q|q' >< q'| &= < q|, \text{and} \\ \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} < k|k' >< k'| &= < k| \end{aligned} \tag{2.12}$$

If the relations Eq.(2.12) are the bra forms ($<|$), obtain the ket forms ($|>$).
We can now conclude that:

$$\begin{aligned} < q|q' > &= \delta(q - q'), \text{and} \\ < k|k' > &= 2\pi\, \delta(k - k'), \end{aligned} \tag{2.13}$$

where for example, the symbolic function $\delta(q - q')$ vanishes everywhere except precisely at the point $q = q'$, and its value there cannot be finite, since it is integrated on a measure of zero and still we get a non-zero value for the integral. Furthermore , we compare the right-hand-side and the left-hand-side of Eq.(2.12) and conclude that the integral of $\delta(q-q')$ with respect to q' over any interval including q is unity. We can get an integral representation of this symbolic function:

$$< q|q' >= \delta(q - q') =< q|1_{op}|q' >= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(q-q')}. \tag{2.14}$$

Show that delta function is a symmetric function (Hint: change the integration variable $k \to -k$ and interpret the result.

Exercise: Show that if one defines the vectors $|k>$ with a different normalization, such that

$$|k>=\sqrt{\frac{L}{2\pi}}\,|n>\,, \tag{2.15}$$

then the plane wave functions and orthogonality relations become

$$\begin{aligned} <q|k> &= \frac{e^{ikq}}{\sqrt{2\pi}} \\ <q|q'> &= \delta(q-q') \\ <k|k'> &= \delta(k-k'). \end{aligned} \tag{2.16}$$

It is only a matter of taste to choose the set (2.13) or (2.16).

Even though we found the integral representation of the delta function the integral in Eq.(2.14) is formally trivial but with the indeterminate result that $\frac{sin\infty(q-q')}{\pi(q-q')}$. Obviously, this way does not work. What is the source of the problem? It is because very high k values allowed in the integral cause problem [6]. We should not allow k to take on infinitely large values. Thus we need to insert a convergence factor which is unity for small k values and vanishes for large k values. Let's call this convergence factor $K(\epsilon k)$ where ϵ is a small, positive, real parameter. We demand that for fixed ϵ, as $k \to \infty$ the convergence factor should vanish, and for fixed k as $\epsilon \to 0^+$ the convergence factor should go to unity. With this factor inserted, we first take the integral, (we write $Q = q - q'$ for simplicity)

$$\delta(Q) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} K(\epsilon k) e^{ikQ}. \tag{2.17}$$

and after that let $\epsilon \to 0^+$. There are many possibilities for $K(\epsilon k)$. A few are as follows:

$K(\epsilon k) = \Theta(\frac{1}{\epsilon} - |k|)$ where Θ is a unit step function,

$K(\epsilon k) = e^{-\epsilon|k|}$, and

$K(\epsilon k) = \frac{sin\frac{\epsilon k}{2}}{\frac{\epsilon k}{2}}$

As an example, let us take for the convergence factor $K(\epsilon k) = e^{-\epsilon|k|}$, to supress the high k values in the integral, take the integral and then let $\epsilon \to 0^+$. Thus let us consider the integral I defined as

$$I = \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{-\epsilon|k|}\, e^{ikQ} \tag{2.18}$$

The integral is again easy and we obtain the result :

$$I = \lim_{\epsilon\to 0^+} \frac{1}{2\pi}\frac{2\epsilon}{\epsilon^2 + Q^2}. \tag{2.19}$$

Note that if $Q \neq 0$, then integral I vanishes after we take the limit. If, however, $Q = 0$ then as ϵ approaches zero, the integral $I \to \infty$. So far fine. Next we must take the integral over Q and see what we get:

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dQ \frac{2\epsilon}{\epsilon^2 + Q^2} = \frac{1}{2\pi} 2tan^{-1}(\frac{Q}{\epsilon})\Big|_{-\infty}^{+\infty} = 1. \tag{2.20}$$

Hence:

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi}\frac{2\epsilon}{\epsilon^2 + Q^2} = \delta(Q). \tag{2.21}$$

and:

$$\int_{-\infty}^{+\infty} dQ\delta(Q) = 1 \tag{2.22}$$

Thus we see that the symbolic function $\delta(Q)$ really is operational under an integral sign. There are many possible realizations of the delta function depending on the choice of the convergence factor $K(\epsilon k)$. All we require from $\delta(Q)$ is:

1. $\delta(Q) = 0$ for $Q \neq 0$.
2. $\delta(Q) \to \infty$ for $Q = 0$.
3. $\int_{-\infty}^{+\infty} dQ\ \delta(Q) = 1$.

Exercise: show that the following function is a δ-function (that is, it satisfies the three requirements stated above):

$$\lim_{\epsilon\to 0^+} \frac{1}{\sqrt{\pi}\,\epsilon} \exp\left[-\frac{Q^2}{\epsilon^2}\right] \tag{2.23}$$

As a result of these properties, delta function is a sampling function:

$$\int_{-\infty}^{+\infty} dQ\delta(Q)f(Q) = f(0). \tag{2.24}$$

It samples that value of $f(Q)$ for which the argument of $\delta(Q)$ vanishes, which in this case is $Q = 0$.
Let us first find the Fourier transforms of a few functions.

Problem set: 2

1. A rectangular pulse $|f>$of height H extends from $x=-a/2$ to $x=+a/2$.
a) Find its Fourier transform $f(k)$, this integral is easy.
Answer:

$$< k|f >= \int_{-\infty}^{+\infty} dq < k|q >< q|f >= aH \left(\frac{sin\frac{ak}{2}}{\frac{ak}{2}} \right)$$

b) Plot this function , as a function of its argument $\frac{ak}{2}$.
c) We know of course what the Fourier transform of $f(k)$ is, it is our $f(q)$ of course. But let us take the integrals explicitly, and these integrals involve more work.
Answer:

$$< q|f >= H\Theta(\frac{a}{2} - |q|),$$

where $\Theta(s)$ is the unit step function. But to arrive this result use(for A real):

$$\int_{-\infty}^{+\infty} dy \frac{e^{iAy}}{y} = i\pi \left(\Theta(A) - \Theta(-A)\right)$$

which can easily be shown using the techniques of contour integration.

2. Find the Fourier transform $f(\omega)$ of the following signals. And also take the inverse transforms to get back to the function you started with. Take integrals explicitly.

$f_1(t) = e^{i\omega_o t}$,

$f_2(t) = sin\omega_o t$,

$f_3(t) = cos\omega_o t$,

$f_4(t) = \delta(t)$,

$f_5(t) = \Theta(t)$ (hint: insert a convergence factor $\lim_{\epsilon\to 0^+} e^{-\epsilon t}$ into the integral. For the inverse transform best is to use a contour integration)

3. Show the following operational properties of the delta function: (Apply the functional $\int_{-\infty}^{+\infty} dq \ldots$)

$\delta(q) = \delta(-q)$,

$\delta(aq) = \frac{1}{|a|}\delta(q)$, a is real,

If $h(q)$ has simple zeros at $\{q_n\}$, that is $h(q_n) = 0$ but $h'(q_n) \neq 0$, then
$\delta[h(q)] = \sum_n \frac{\delta(q-q_n)}{|h'(q_n)|}$
Example, compute $\int_{-\infty}^{+\infty} dq\delta(1-q^2)\, f(q)$

$q\delta(q) = 0$

$f(q)\delta(q-a) = f(a)\delta(q-a)$

$\int_{-\infty}^{+\infty} dq f(q) \left[\frac{d}{dq}\delta(q)\right] = -f'(0)$ hint:integrate by parts

$\int_a^b dq\delta(q-c)\delta(q-d) = \delta(c-d),$ where $a < (c,d) < b$.

2.2 Convolution

a) Consider the vectors $|f_1>$ and $|f_2>$. The vector $|F>$ is defined such that $F(q) = f_1(q) f_2(q)$. The question is to compute the Fourier transform $<k|F>$. We proceed step by step:

$$
\begin{aligned}
<k|F> &= <k|1_{op}|F> = \int_{-\infty}^{+\infty} dq <k|q><q|F> \qquad (2.25) \\
&= \int dq e^{-ikq} <q|f_1><q|f_2> \\
&= \int dq e^{-ikq} <q|1_{op}|f_1><q|1_{op}|f_2> \\
&= \int dq e^{-ikq} \int \frac{dk_1}{2\pi} <q|k_1><k_1|f_1> \int \frac{dk_2}{2\pi} <q|k_2><k_2|f_2>
\end{aligned}
$$

we now change the order of integration

$$
= \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} <k_1|f_1><k_2|f_2> \int dq e^{iq(k_1+k_2-k)}
$$

the integral over q is 2π times the delta function

$$
= \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} <k_1|f_1><k_2|f_2> 2\pi\delta(k_1+k_2-k)
$$

if we set $k_2 = k - k_1$ we get

$$
F(k) = \int \frac{dk_1}{2\pi} f_1(k_1) f_2(k-k_1)
$$

or if we set $k_1 = k - k_2$

$$
F(k) = \int \frac{dk_2}{2\pi} f_1(k-k_2) f_2(k_2).
$$

We now write this important result again:

$$\text{If } F(q) = f_1(q) f_2(q) \text{ then,} \tag{2.26}$$

$$F(k) = \int \frac{dk'}{2\pi} f_1(k') f_2(k-k') \tag{2.27}$$

$$= \int \frac{dk'}{2\pi} f_1(k-k') f_2(k') ,$$

and say, $F(k)$ is the convolution of $f_1(k)$ and $f_2(k)$, (where the convolution is defined by the integral in Eq.(2.27)) and denote this operation also, by a special "convolution symbol" $\circ$, that is,

$$F(k) = (f_1 \circ f_2)\ (k) = (f_2 \circ f_1)\ (k) .$$

b) Now the other way around: Consider the vectors $|f_1>$ and $|f_2>$. The vector $|F>$ is defined such that $F(k) = f_1(k) f_2(k)$. The question is to compute the Fourier transform $<q|F>$. Proceed step by step as in Eq.(2.25) and obtain the result:

$$\text{If } F(k) = f_1(k) f_2(k) \text{ then,} \tag{2.28}$$

$$F(q) = \int dq' f_1(q') f_2(q-q') \tag{2.29}$$

$$= \int dq' f_1(q-q') f_2(q') ,$$

and say, $F(q)$ is the convolution of $f_1(q)$ and $f_2(q)$.

We can represent the function $f_2(q-q')$ in terms operators acting on $f_2(q')$ as follows. Let us define the *translation* operator $\mathcal{T}_a$ and the *mirror imaging* operator $\mathcal{MI}$ such that:

$$\mathcal{T}_a\, f(q') = f(q'-a)$$

$$\mathcal{MI}\, f(q') = f(-q') . \tag{2.30}$$

Obviously these operators have the following properties:

$$\mathcal{T}_a\,\mathcal{T}_b = \mathcal{T}_b\,\mathcal{T}_a = \mathcal{T}_{a+b}$$

$$\mathcal{MI}\;\mathcal{MI} = 1_{op}\,. \tag{2.31}$$

It can be shown that (see the problem below)

$$\mathcal{T}_a\,\mathcal{MI}\,\mathcal{T}_a = \mathcal{MI}\,. \tag{2.32}$$

Thus we can now represent the function $f(q-q')$ as

$$f(q-q') = \mathcal{T}_q\;\mathcal{MI}\;f(q') = \mathcal{MI}\;\mathcal{T}_{-q}\;f(q')\,. \tag{2.33}$$

Thus the convolution shown in Eq.(2.29) involves the overlaps of f_1 and the mirror-imaged and translated f_2, (of course, the indices 1 and 2 can be interchanged).

Problem: Consider a function $f(q')$ consisting of two straight line segments AB and BC where $A = (-1,0)$, $B = (0,H)$ and $C = (2,0)$. Show that the AB segment is $y_1 = H(q'+1)$ and the BC segment is $y_2 = H(-\frac{q'}{2}+1)$.

a)

- Where are the new ABC after a mirror-image operation on f? (hint: $A \to A_{\mathcal{MI}} = (1,0)$). Show that the segments $A_{\mathcal{MI}}B_{\mathcal{MI}}$ and $B_{\mathcal{MI}}C_{\mathcal{MI}}$ are given by

$$y_{1,\,\mathcal{MI}} = y_1(-q') \;\text{ and }\; y_{2,\,\mathcal{MI}} = y_2(-q')\,.$$

- Where are the new ABC after a translation operation by +3 units on mirror-imaged f ?(hint: $A \to A_{\mathcal{T}_3\,\mathcal{MI}} = (4,0)$). Show that

$$y_{1,\,\mathcal{T}_3\mathcal{MI}} = y_1(-q'+3) \;\text{ and }\; y_{2,\,\mathcal{T}_3\mathcal{MI}} = y_2(-q'+3)\,.$$

Now we change the order of operations:

- Where are the new ABC after a translation operator by +3 units acts on f ?(hint: $A \to A_{\mathcal{T}_3} = (2,0)$). Show that

$$y_{1,\,\mathcal{T}_3} = y_1(q'-3) \;\text{ and }\; y_{2,\,\mathcal{T}_3} = y_2(q'-3)\,.$$

- Where are the new ABC after a mirror-image operation acts on translated f? (hint: $A \rightarrow A_{\mathcal{MI},\mathcal{T}_3} = (-2,0)$). Show that

$$y_{1,\,\mathcal{MI}\,\mathcal{T}_3} = y_1(-q'-3) \;\text{ and }\; y_{2,\,\mathcal{MI}\,\mathcal{T}_3} = y_2(-q'-3)\,.$$

b) Show that

$$\mathcal{T}_{-6}\,\mathcal{T}_3\,\mathcal{MI} = \mathcal{T}_{-3}\,\mathcal{MI} = \mathcal{MI}\,\mathcal{T}_3\,.$$

and by replacing $+3$ by a general displacement a, obtain the relation 2.32.

c) Application:

Consider an analog time signal $S(t)$ with Fourier transform $S(\omega)$. (For this example, $q \rightarrow t$ and $k \rightarrow \omega$). This signal is observed from time $t = -\frac{T}{2}$ to $t = \frac{T}{2}$. So the information we have is not that of $S(t)$ but that of a modified(windowed) signal $S_{Windowed}(t) = S(t)W_R(t)$. Here the rectangular window function $W_R(t)$ and its Fourier transform $\hat{W}_R(\omega)$ are given by (see problem 1):

$$W_R(t) = \Theta\left(\frac{T}{2} - |t|\right) \tag{2.34}$$

$$\hat{W}_R(\omega) = T\frac{sin\frac{\omega T}{2}}{\frac{\omega T}{2}} \tag{2.35}$$

Plot $\hat{W}_R(\omega)$. The Fourier transform of the observed signal is not $S(\omega)$ but $S_{Windowed}(\omega)$ and through convolution theorem is given by:

$$S_{Windowed}(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} S(\omega')\hat{W}_R(\omega - \omega') \tag{2.36}$$

Note that the function $\hat{W}_R(\omega - \omega')$, as a function of ω', is just the shape of $\hat{W}_R(\omega')$ mirror imaged and then translated (shifted) by ω,

$$\hat{W}_R(\omega - \omega') = \mathcal{T}_\omega\;\mathcal{MI}\;\hat{W}_R(\omega') \tag{2.37}$$

by virtue of the Eq.(2.32). If we express the integrals in terms of the rotational frequency:

$$S_{Windowed}(f) = \int_{-\infty}^{+\infty} df' S(f')\left\{T\frac{sin\pi T(f-f')}{\pi T(f-f')}\right\}. \tag{2.38}$$

Now if the true signal has only one frequency, $S(f) = A\delta(f - f_o)$ then

$$S_{Windowed}(f) = A\left\{T\frac{sin\pi T(f - f_o)}{\pi T(f - f_o)}\right\}, \tag{2.39}$$

which means we observe not a single frequency, but a distribution of frequencies peaked around f_o . Show that the half-width of the principle lobe of this distribution is $\frac{1}{T}$ and hence resolving power will increase if we increase T, the time of observation of the signal. Hence we cannot resolve frequencies in the band $f_o \pm \Delta f$ where $\Delta f = \frac{1}{T}$.

If the true signal has more than one frequency, each will be smeared out by the convolving window . And if the signal contains a distribution of frequencies, each frequency band will be convolved by the window.
In short, the shorter the observation time, the less information we can get out of the signal.

2.3 Sampling

a) We first consider the Fourier transform of periodic functions $f(q) = f(q+nL)$. Since the function is periodic it can be expanded in a Fourier series:

$$f(q) = \sum_{n=-\infty}^{+\infty} c_n \left(\frac{e^{ink_oq}}{\sqrt{L}}\right) \text{ , where } k_o = \tfrac{2\pi}{L} \text{ ,} \tag{2.40}$$

We now compute the Fourier transform of $f(q)$:

$$\begin{aligned} < k|f > &= \int dq < k|q >< q|f > \\ &= \int dq e^{-ikq} \sum_{n=-\infty}^{+\infty} c_n \left(\frac{e^{ink_oq}}{\sqrt{L}}\right) \\ < k|f > &= \sum_{n=-\infty}^{+\infty} \frac{c_n}{\sqrt{L}} 2\pi\delta(k - nk_o). \end{aligned} \tag{2.41}$$

Note that because of the presence of the term c_n in the sum , the fourier transform of a periodic function is , in general, *not* periodic : $f(k) \neq f(k + nk_o)$.

A periodic function which has also a periodic transform, however, is the *infinite impulse train* $I_L(q)$:

$$I_L(q) = \sum_{n=-\infty}^{+\infty} \delta(q - nL). \tag{2.42}$$

Problem 1 i) Show that this is indeed a periodic function, that is, $I_L(q) = I_L(q + nL)$.
ii) Expand the infinite impulse train in fourier series, since it is periodic:

$$I_L(q) = \sum_{n=-\infty}^{+\infty} c_n \left(\frac{e^{ink_o q}}{\sqrt{L}}\right) \text{, where } k_o = \tfrac{2\pi}{L} \text{ ,} \tag{2.43}$$

and the coefficients c_n are computed for one periodic cell $q_o \leq q \leq q_o + L$,

$$c_n = \int_{q_o}^{q_o+L} dq \left(\frac{e^{-ink_o q}}{\sqrt{L}}\right) \sum_{m=-\infty}^{+\infty} \delta(q - mL) = \frac{1}{\sqrt{L}} . \tag{2.44}$$

The result (2.44) follows because if, say, $(M-1)L < q_o < ML$, then in the interval $[q_o, q_o + L]$ only one of the delta functions, in this case $\delta(q - ML)$ contributes to the integral. Thus all the fourier coefficients $c_n = \frac{1}{\sqrt{L}}$ are independent of n. The infinite impulse train can thus also be represented as

$$I_L(q) = \sum_{n=-\infty}^{+\infty} \delta(q - nL) = \frac{1}{L} \sum_{n=-\infty}^{+\infty} e^{i\frac{2\pi}{L} nq} . \tag{2.45}$$

The fourier transform can now easily be found from Eq.(2.41), we just need to insert the value of c_n from Eq.(2.44). Show that

$$I_L(k) = k_o \sum_{n=-\infty}^{+\infty} \delta(k - nk_o), \text{ where } k_o = \tfrac{2\pi}{L} \text{ .} \tag{2.46}$$

iii) Hence the fourier transform of the infinite impulse train is also periodic, $I_L(k) = I_L(k + nk_o)$.
iv) Rewrite the formulas of this section when $(q \to t,\ k \to \omega,\ L \to T,\ k_o \to \omega_o = \frac{2\pi}{T})$ to familiarize yourself with the time-frequency expressions.

b) Sampling theorem Consider a continuous-time analog signal $f_a(t)$. We sample it at times $t = nT_s$ with the infinite impulse train $I_{T_s}(t)$. The result is the sampled analog signal $f_{a,s}(t)$:

$$f_{a,s}(t) = f_a(t)\ I_{T_s}(t). \tag{2.47}$$

The fourier transform of the sampled signal involves the convolution of the fourier transforms of $f_a(t)$ and $I_{T_s}(t)$, (see Eqs.(2.26) and (2.27)) :

$$f_{a,s}(\omega) = \int \frac{d\omega'}{2\pi} f_a(\omega')\, I_{T_s}(\omega - \omega'). \tag{2.48}$$

Show that the result of the convolution integral is:

$$f_{a,s}(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} f_a(\omega - n\omega_s) \text{ where } \omega_s = \tfrac{2\pi}{T_s} \tag{2.49}$$

Hence, $f_{a,s}(\omega)$ is the periodically repeated and scaled (i.e., multiplied with $\frac{1}{T_s}$) replica of $f_a(\omega)$. Draw a figure of $f_a(\omega)$ and $f_{a,s}(\omega)$.

Let us assume that our signal is band-limited, i.e., let ω_{max} be the highest angular frequency contained in the spectrum of our analog signal:

$$f_a(\omega) = \begin{cases} f_a(\omega) & \text{if} \quad |\omega| < \omega_{max}\ , \\ 0 & \text{if} \quad |\omega| > \omega_{max}. \end{cases} \tag{2.50}$$

In that case, show that if $\omega_s > 2\omega_{max}$ adjacent replicas do not overlap, and if $\omega_s < 2\omega_{max}$ adjacent replicas overlap. If $\omega_s > 2\omega_{max}$ we can write the fourier transform of the analog signal in terms of the sampled-analog signal as

$$f_a(\omega) = f_{a,s}(\omega)\, T_s\, \Theta\left(\omega_{max} - |\omega|\right) \text{ where } \omega_s > 2\omega_{max}\ . \tag{2.51}$$

It is now obvious how to recover the analog-continuous time signal $f_a(t)$ from its samples. Use a low-pass filter to eliminate the spectral components with frequencies $|\omega| > \omega_{max}$ with a gain (i.e., multiply with) T_s . Also to make sure that the analog signal has no frequency components higher than $\omega_s/2$, first use an antialiasing filter on the analog signal before the sampling operation as was discussed at the end of the section on aliasing 1.3.
We can now transform Eq.(2.51) to the time domain. Since on the right-hand-side we have a product of frequency functions, their fourier transform involves the convolution of time functions, (see Eqs.(2.28) and (2.29)):

$$
\begin{aligned}
f_a(t) &= T_s \int dt' f_{a,s}(t') \left\{ \frac{\omega_{max}}{\pi} \frac{sin\, \omega_{max}(t-t')}{\omega_{max}(t-t')} \right\}, \\
&= T_s \int dt' \left[f_a(t')\, I_{T_s}(t') \right] \left\{ \frac{\omega_{max}}{\pi} \frac{sin\, \omega_{max}(t-t')}{\omega_{max}(t-t')} \right\} \\
&= T_s \int dt' \left[f_a(t') \sum_{n=-\infty}^{+\infty} \delta(t'-nT_s) \right] \left\{ \frac{\omega_{max}}{\pi} \frac{sin\, \omega_{max}(t-t')}{\omega_{max}(t-t')} \right\} \\
&= \frac{2\omega_{max}}{\omega_s} \sum_{n=-\infty}^{+\infty} f_a(nT_s) \frac{sin\, \omega_{max}(t-nT_s)}{\omega_{max}(t-nT_s)}. \qquad (2.52)
\end{aligned}
$$

In the equation above the term in the curly brackets $\{\ldots\}$ is the fourier transform of the box function in the frequency domain $\Theta\left(\omega_{max} - |\omega|\right)$. Thus, an analog signal can be reconstructed exactly from its samples if the sampling rate is at least twice the highest frequency component present in the signal.

2.4 Uncertainty Products

There is an important aspect of Fourier transforms that we must go over before proceeding further. It is related to the widths of the transform pairs. Since we have a complete set of basis vectors, we can define operators in this linear vector space by their actions on the basis vectors and their matrix elements. Let us define the operator k_{op} such that vectors $|k>$ are the eigenvectors of this operator:

$$
< k_1|k_{op} = k_1 < k_1|. \qquad (2.53)
$$

We can write the matrix elements of the operator k_{op} in k-space as follows:

$$
< k_1|k_{op}|k_2 >= k_1 < k_1|k_2 >= k_2 < k_1|k_2 > \qquad (2.54)
$$

where the last equality followed because $< k_1|k_2 >$ is a delta function. Since in Eq.(2.54) $< k_1|$ is an arbitrary vector , we conclude that

$$
k_{op}|k_2 >= k_2|k_2 > . \qquad (2.55)
$$

Since , by definition, the Hermitian conjugate of Eq.(2.55) is

$$< k_2|(k_{op})^\dagger = k_2^* < k_2| = k_2 < k_2|, \tag{2.56}$$

where the last equality follows because the eigenvalue k_2 is real. We conclude upon comparing with Eq.(2.53)that the operator k_{op} is Hermitian

$$k_{op} = (k_{op})^\dagger. \tag{2.57}$$

What is the action of this operator on the q space vectors ? Let us compute

$$\begin{aligned} < q_1|k_{op} &= < q_1|1_{op}k_{op} = \int \frac{dk_1}{2\pi} e^{iq_1k_1} k_1 < k_1| \\ &= \frac{1}{i}\frac{\partial}{\partial q_1} < q_1|. \end{aligned} \tag{2.58}$$

In a similar way we define a q_{op} for which the q-basis vectors are the eigenvectors:

$$< q_1|q_{op} = q_1 < q_1|. \tag{2.59}$$

Exercise: Show that the q_{op} is Hermitian,

$$q_{op} = (q_{op})^\dagger. \tag{2.60}$$

What is the action of this operator on the k space vectors ? Let us compute

$$\begin{aligned} < k_1|q_{op} &= < k_1|1_{op}q_{op} = \int dq_1 e^{-iq_1k_1} q_1 < q_1| \\ &= -\frac{1}{i}\frac{\partial}{\partial k_1} < k_1|. \end{aligned} \tag{2.61}$$

The operators q_{op} and k_{op} do not commute. Let $|\Phi>$ be an arbitrary vector in this vector space,and let's compute

$$\begin{aligned} &< q_1| \left(q_{op}k_{op} - k_{op}q_{op}\right)|\Phi> \\ &= q_1\frac{1}{i}\frac{\partial}{\partial q_1} < q_1|\Phi> - \frac{1}{i}\frac{\partial}{\partial q_1} q_1 < q_1|\Phi> \\ &= -\frac{1}{i} < q_1|\Phi> . \end{aligned} \tag{2.62}$$

Since $< q_1|$ and $|\Phi >$ are arbitrary vectors, we conclude from Eq.(2.62) that the commutator is given by:

$$[\, q_{op} \,, k_{op} \,] \equiv q_{op}k_{op} - k_{op}q_{op} = \, i\, 1_{op} = \, i \; . \tag{2.63}$$

Problem If $\langle f|f\rangle = 1$ then $|f(q)|^2$ can be interpreted as a probability distribution in q. Define the mean and the mean-square values as:

$$\begin{aligned} \overline{q} &\equiv \langle f|q_{op}|f\rangle \\ \overline{k} &\equiv \langle f|k_{op}|f\rangle \\ \overline{q^2} &\equiv \left\langle f|q_{op}^2|f\right\rangle \\ \overline{k^2} &\equiv \left\langle f|k_{op}^2|f\right\rangle \end{aligned} \tag{2.64}$$

The uncertainties Δq and Δk in q and k , respectively, are defined as root-mean square deviations $\Delta q = \left[\overline{(\Delta q)^2}\right]^{\frac{1}{2}}$ and $\Delta k = \left[\overline{(\Delta k)^2}\right]^{\frac{1}{2}}$, where the mean square deviations are:

$$\begin{aligned} \overline{(\Delta q)^2} &= \overline{q^2} - \overline{q}^2 \\ \overline{(\Delta k)^2} &= \overline{k^2} - \overline{k}^2 \end{aligned} \tag{2.65}$$

Consider the vector $|G\rangle \equiv \left(q_{op}^{fl} + i\lambda k_{op}^{fl}\right)|f>$ where λ is a real scalar, and the fluctuation operators are:

$$\begin{aligned} q_{op}^{fl} &= q_{op} - \overline{q} \; , \qquad (2.66) \\ k_{op}^{fl} &= k_{op} - \overline{k} \; . \end{aligned}$$

Show that

$$\begin{aligned} \overline{(\Delta q)^2} &= \; < f|\left(q_{op}^{fl}\right)^2|f > \; , \qquad (2.67) \\ \overline{(\Delta k)^2} &= \; < f|\left(k_{op}^{fl}\right)^2|f > \; . \end{aligned}$$

Compute the norm $\langle G|G\rangle = N_\lambda$:
(Hint: the operators q_{op} and k_{op} are hermitian and $\langle G| =< f| \left(q_{op}^{fl} - i\lambda k_{op}^{fl}\right)$),

$$N_\lambda = (\Delta q)^2 + \lambda^2 (\Delta k)^2 + i\lambda < f| \left[q_{op}^{fl}\,,\, k_{op}^{fl}\right] |f>\,, \tag{2.68}$$

use Eq.(2.63) and evaluate the commutator

$$= (\Delta q)^2 + \lambda^2 (\Delta k)^2 - \lambda\,.$$

Hence N_λ considered as a function of λ is a parabola. Show that its minimum occurs at λ_o given by $\lambda_o = \frac{1}{2} (\Delta k)^{-2}$. Since the norm of a vector is positive semidefinite, show that the condition $N_{\lambda_o} \geq 0$ leads to the statement of the uncertainty relation:

$$\Delta q\; \Delta k \geq \frac{1}{2}. \tag{2.69}$$

Express and interpret this important result in words.

If we are working in the (x, p) spaces (distance, wavenumber) then:

$$\Delta x\; \Delta p \geq \frac{1}{2}. \tag{2.70}$$

or if we are working in the (t, ω) spaces (time, frequency) then:

$$\Delta t\; \Delta w \geq \frac{1}{2}. \tag{2.71}$$

This says that we cannot arbitrarily reduce the widths of the fourier transform pairs. Their product is above (or equal to) a fixed minimum ($= \frac{1}{2}$). For example a signal at a definite frequency ($\Delta\omega = 0$) must last infinitely long in time. Or a a signal that is an impulse in time ($\Delta t = 0$) has all the frequencies in its spectrum .
Likewise, a signal at a definite wavelength ($\Delta p = 0$) must have infinite extent. Or a signal that is an impulse in space ($\Delta x = 0$) has all the wavelengths in its spectrum.

Problem set: 3

1. State of minimum uncertainty Consider the q-space function

$$< q|\psi_o >= \frac{1}{\pi^{1/4}} e^{-q^2/2} \ . \tag{2.72}$$

Show that
a) State is normalized to unity.

$$< \psi_o|\psi_o >= 1 \ . \tag{2.73}$$

b) Mean q vanishes,

$$\overline{q} =< \psi_o|q_{op}|\psi_o >= 0 \ . \tag{2.74}$$

c) Mean square q-deviation is

$$\begin{aligned} (\Delta q)^2 &= \overline{q^2} - \overline{q}^2 \\ \overline{q^2} =< \psi_o|q_{op}^2|\psi_o > &= \frac{1}{2} \end{aligned} \tag{2.75}$$

d) Fourier transform $\psi_o(k)$ is given by

$$< k|\psi_o >= \sqrt{2}\, \pi^{1/4} e^{-k^2/2} \ . \tag{2.76}$$

hint: use the result of HW2-Problem 4:

$$I = \int_{-\infty}^{+\infty} dq\, e^{-ikq}\, e^{-\frac{q^2}{2\sigma^2}} = \sqrt{2\pi}\, \sigma\, e^{-\frac{k^2\sigma^2}{2}} \ . \tag{2.77}$$

e) Mean k vanishes,

$$\overline{k} =< \psi_o|k_{op}|\psi_o >= 0 \ . \tag{2.78}$$

f) Mean square k-deviation is

$$\begin{aligned} (\Delta k)^2 &= \overline{k^2} - \overline{k}^2 \\ \overline{k^2} =< \psi_o|k_{op}^2|\psi_o > &= \frac{1}{2} \end{aligned} \tag{2.79}$$

Thus the state $|\psi_o >$ has the minimumu uncertainty product:

$$\Delta q\, \Delta k = \frac{1}{2} \ . \tag{2.80}$$

It can be shown, through a variational calculation that, the Gaussian form is unique in having the minimum uncertainty product.

2. Algebraic properties of the Fourier Transform (FT) If $f(t)$ has FT $\hat{f}(\omega)$, then show that:

	$F(t)$	$\hat{F}(\omega)$
Translation	$f(t-t_o)$	$e^{-i\omega t_o}\hat{f}(\omega)$
Modulation	$e^{i\omega_o t}f(t)$	$\hat{f}(\omega-\omega_o)$
Scaling	$f(t/s)$	$\|s\|\,\hat{f}(s\,\omega)$
Complex conj.	$f^*(t)$	$\hat{f}^*(-\omega)$
Real	$f(t)=f^*(t)$	$\hat{f}(-\omega)=\hat{f}^*(\omega)$

Chapter 3

Discrete Fourier Transform

3.1 N^{th} roots of unity

We want to solve the equation

$$\mathcal{Z}^N - 1 = 0, \tag{3.1}$$

where complex roots are allowed. Since this is a polynomial it must have N roots. If we denote $+1 = e^{2\pi i}$ then roots are easily found:

$$z_k = e^{i\frac{2\pi}{N}k}, \text{ where } k = 0, 1, \ldots, N-1 \ . \tag{3.2}$$

So, the roots are all distinct and given by $z_0, \ z_1, \ \ldots, z_{N-1}$. The root z_1 is called the primitive root of unity because all other roots can be generated by z_1, that is, $z_1^k = z_k$ with $z_1^N = z_0$.
In the following problem set you will review the properties of N^{th} roots of unity.

Problem set: 4

1. Below are simple exercises:

Exercise 1: Check that $z_k^N = +1$ for $k = 0, 1, \ldots, N-1$.

Exercise 2: Show the roots as vectors in the complex plane for $N = 5$ and $N = 6$.

Exercise 3: Show that the N roots are the vertices of a regular polygon of N sides inscribed in a circle of unit radius, with center at the origin, with one vertex at the real number $+1$.

Exercise 4: Show that z_1 generates all the roots, $z_1^k = z_k$ with $z_1^N = z_0$.

Exercise 5: Show that if $z_1^\ell = z_\ell$ is also a primitive root of unity, that is we can relabel the roots such that $z_\ell \to z_{1,new}$ and $z_{1,new}$ generates all the other roots, then (ℓ, N) are relative primes (do not have a common divisor). Consider the roots generated by z_2, z_3, z_4 for $N = 5$ and the roots generated by z_2, z_3, z_4, z_5 for $N = 6$. For each case find the primitive roots of unity, and relabel the roots accordingly.

Exercise 6: Show that the multiplicative inverse of z_k is also one of the roots and that $(z_k)^{-1} = z_{N-k}$.

Exercise 7: Show that the complex conjugate of a root is given by its inverse, $(z_k)^* = z_{-k} = (z_k)^{-1} = z_{N-k}$. Note that by defining z_{-k} as z_{N-k}, we are using modulo(N) numbers to index the roots.

2. The N^{th} roots of unity satisfy the following *sum rules*:

$$\sum_{k=0}^{N-1} (z_k)^m = \begin{cases} 0 & \text{for } m = 1, 2, \ldots, N-1 \ , \\ N & \text{for } m = 0,\ N \ . \end{cases} \tag{3.3}$$

We now want to prove this relation.
The polynomial $\mathcal{Z}^N - 1$ can be factored out in terms of the roots as:

$$\mathcal{Z}^N - 1 = (\mathcal{Z} - z_0)(\mathcal{Z} - z_1)\ldots(\mathcal{Z} - z_{N-1})$$

i) Divide by $\mathcal{Z}^N$ and take the logarithm of both sides and arrange to obtain:

$$ln\left(1 - \frac{1}{\mathcal{Z}^N}\right) = ln\left(1 - \frac{z_0}{\mathcal{Z}}\right) + ln\left(1 - \frac{z_1}{\mathcal{Z}}\right) + \ldots + ln\left(1 - \frac{z_{N-1}}{\mathcal{Z}}\right).$$

ii) Now expand the ln functions:

$$ln(1-u) = -\sum_{m=1}^{\infty} \frac{u^m}{m} = -u - \frac{u^2}{2} - \frac{u^3}{3} - \ldots,$$

and compare the powers of $\frac{1}{\mathcal{Z}}$ on both sides of the equation. Left hand side starts with $\frac{1}{\mathcal{Z}^N}$ so on the right hand side the coefficients of $\frac{1}{\mathcal{Z}^m}$ with $m = 1, 2, \ldots, N-1$ must vanish. Hence obtain the result Eq.(3.3).

3.2 Basis vectors in $\mathcal{C}^N$

A vector space of N dimensions where the components of a vector are allowed to be complex is denoted by $\mathcal{C}^N$. One obvious set of orthonormal basis vectors are $|q_n>$ where the n^{th} component is unity and others are zero. Clearly.

$$< q_n|\, q_m >= \delta_{n\,m} \quad , \tag{3.4}$$

where the index n runs from $n = 0$ to $n = N-1$. For later convenience, we impose cyclical periodicity by demanding $|q_0 >\equiv |q_N >$. Thus $|q_n >= |q_{n+mN} >$ where m is any integer (zero, positive or negative) and period is obviously N. We now construct a new set of basis vectors $|k>$, $k = 0, 1, \ldots, N-1$, which are linear combinations of the $|q_n>$s and where the coefficients are powers of the N^{th} roots of unity,

$$|k> \;=\; \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} z_n^k \,|q_n> \;=\; \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ z_1^k \\ z_2^k \\ . \\ . \\ . \\ z_{N-1}^k \end{pmatrix} , \tag{3.5}$$

and the components of the conjugate vector $< k|$ are given by the complex conjugates :

$$\begin{aligned} < k| \;&=\; \frac{1}{\sqrt{N}} \{ z_0^{k*},\; z_1^{k*},\; \ldots,\; z_{N-1}^{k*} \} \;, \\ &=\; \frac{1}{\sqrt{N}} \{ z_0^{-k},\; z_1^{-k},\; \ldots,\; z_{N-1}^{-k} \} \end{aligned} \tag{3.6}$$

where we used the results of the Exercise 7 . Let us compute the overlap $< k_i|k_j >$,

$$\begin{aligned} < k_i|k_j > \;&=\; \frac{1}{N} \sum_{n,\,m} z_n^{k_i\,*} z_m^{k_j} < q_n|\, q_m > \\ &=\; \frac{1}{N} \sum_n z_n^{k_j - k_i} \\ &=\; \delta_{ij} \;. \end{aligned} \tag{3.7}$$

To get the last equality above we used the property of the roots given in Eq.(3.3). We may consider the transformation from $|q_n>$ basis to $|k_i>$

basis as affected by a matrix U :

$$< q_n|U \ = < k_n|. \tag{3.8}$$

And the matrix elements of U are then,

$$< q_n|U|q_m >= \ \frac{1}{\sqrt{N}} z_m^{k_n *} \tag{3.9}$$

Exercise 1: Show that the matrix elements of $U^\dagger$ are:

$$< q_a|U^\dagger|q_b >= \frac{1}{\sqrt{N}} z_a^{k_b}. \tag{3.10}$$

Exercise 2: Prove that the U matrix that affects the transormation between $|q>$ and $|k>$ vectors is unitary: $UU^\dagger = U^\dagger U = 1_{op}$.

Problem: Obviously $\sum_{n=0}^{N-1} |q_n \ >< \ q_n| \ = \ 1_{op}$. Use the definition of the U matrix, Eq.(3.8) and the fact that it is unitary , to prove that the vectors $|k_j>$ fully span the vector space,

$$1_{op} = \sum_{n=0}^{N-1} |q_n >< q_n| = \sum_{j=0}^{N-1} |k_j >< k_j|. \tag{3.11}$$

Consider an arbitrary vector $|x>$ in $\mathcal{C}^N$. We can expand this vector either in the q-coordinate system or the k-coordinate system:

$$\begin{aligned} |x> \ &= \ 1_{op}|x >= \sum_{n=0}^{N-1} |q_n >< q_n|x >, \\ &= \ 1_{op}|x >= \sum_{j=0}^{N-1} |k_j >< k_j|x > . \end{aligned} \tag{3.12}$$

If we write the coefficients explicitly,

$$
\begin{aligned}
< q_n|x > &\equiv x(q_n) \\
< k_j|x > &\equiv x(k_j) = \sum_{n=0}^{N-1} \left\{ \frac{1}{\sqrt{N}} e^{-i\frac{2\pi n}{N}k_j} \right\} x(q_n), \\
< q_n|x > &= \sum_{j=0}^{N-1} \left\{ \frac{1}{\sqrt{N}} e^{i\frac{2\pi n}{N}k_j} \right\} x(k_j). \qquad (3.13)
\end{aligned}
$$

Relations given in Eqs.(3.12 and 3.13) constitute the Discrete Fourier Transform pairs.

3.3 DFT in time & frequency

Let us use the variables appropriate for this case: $q \to t$ and $k \to \omega$, and consider a sinusoidal wave $Z(t) = exp(i\tilde{\omega}t)$. It is sampled with frequency f_s, at equally spaced times $t = (0,\ 1,\ \ldots,\ (N-1))T_s$ where $T_s = f_s^{-1}$. It is convenient to measure time in units of T_s and the frequency in units of f_s. So let us define a dimensionless angular frequency ω as $\omega \equiv 2\pi\tilde{f}/f_s$. Hence the expression for our sinusoid becomes:

$$
\begin{aligned}
Z(nT_s) &= exp\left(i\frac{\tilde{\omega}}{\omega_s}2\pi f_s n T_s\right) \\
&= exp\left(i\omega n\right) . \qquad (3.14)
\end{aligned}
$$

We now consider the basis sinusoids in $\mathcal{C}^N$. The basis sinusoids have special values of the dimensionless angular frequency ω: N^{th} roots of unity:

$$
\begin{aligned}
\omega &= \omega_k = \frac{2\pi k}{N}, \quad \text{where } k = 0,\ 1,\ \ldots,\ (N-1) . \\
Z_k(n) &= z_n^k = exp\left(i\frac{2\pi}{N}nk\right) \qquad (3.15)
\end{aligned}
$$

Thus we have obtained the (unnormalized) basis vectors of $\mathcal{C}^N$. It is appropriate now to relabel the basis vectors so as to make various relations clearer (time is measured in terms of T_s and frequency in terms of f_s):

$$\begin{aligned}
|q_n> &\rightarrow |t_n>=|n> \\
|k_j> &\rightarrow |\omega_j> \text{ such that,} \\
<n|m> &= \delta_{nm} \\
<\omega_j|\omega_k> &= \delta_{jk} \\
<n|\omega_k> &= \frac{1}{\sqrt{N}} z_n^k = \frac{1}{\sqrt{N}} z_1^{nk} = \frac{1}{\sqrt{N}} exp\left(i\frac{2\pi}{N}nk\right) \\
1_{op} &= \sum_{n=0}^{N-1} |n><n| = \sum_{k=0}^{N-1} |\omega_k><\omega_k| \ . \qquad (3.16)
\end{aligned}$$

We can easily extend indexing of these vectors to all integers by considering modulo(N) numbers: $n \rightarrow n+mN$ and $k \rightarrow k+mN$. As for the basis vectors we can indicate this periodicity as $|n>=|n+mN>$ and $|\omega_k>=|\omega_{k+mN}>$. We now write the relations given in Eqs.(3.12 and 3.13) for the time and frequency basis vectors.
Consider an arbitrary vector $|x>$ in $\mathcal{C}^N$. We can expand this vector either in the time-coordinate system or the frequency-coordinate system:

$$\begin{aligned}
|x> &= 1_{op}|x>=\sum_{n=0}^{N-1} |n><n|x>, \\
&= 1_{op}|x>=\sum_{k=0}^{N-1} |\omega_k><\omega_k|x> \ . \qquad (3.17)
\end{aligned}$$

If we write the coefficients explicitly,

$$\begin{aligned}
<n|x> &\equiv x(n) \\
<\omega_k|x> &\equiv x(\omega_k) = \sum_{n=0}^{N-1} \left\{ \frac{1}{\sqrt{N}} e^{-in\omega_k} \right\} x(n), \\
<n|x> &= \sum_{k=0}^{N-1} \left\{ \frac{1}{\sqrt{N}} e^{in\omega_k} \right\} x(\omega_k). \qquad (3.18)
\end{aligned}$$

Relations given in Eqs.(3.17 and 3.18) constitute the Discrete Fourier Transform pairs of a function in time and frequency domains. We further extend the definition of the signal in line with those of the basis vectors, and make our signal periodic in N,

$$x(n + mN) \equiv x(n). \tag{3.19}$$

PROPERTIES OF DFT

- **1.** If $x(n)$ is periodic in N so is $x(\omega_k)$. Show that:

$$< \omega_{k+N}|x >=< \omega_k|x > . \tag{3.20}$$

  In particular $x(\omega_0) = x(\omega_N)$

- **2.** Parseval's relation: Show that,

$$< x|x >= \sum_n |x(n)|^2 = \sum_k |x(\omega_k)|^2 \ . \tag{3.21}$$

- **3.** Even functions $x(n)$. By definition:

$$x_{even}(n) = x(-n) = x(N - n) \ . \tag{3.22}$$

  Prove that if $x(n)$ is even, so is $x(\omega_k)$:

$$x(\omega_k) = x(-\omega_k) = x(\omega_{-k}) = x(\omega_{N-k}) \ . \tag{3.23}$$

  Thus evenness is an intrinsic property of $|x >$ and is independent of the coordinate frame used. As an example show explicitly that $< n|x >= cos(\omega n)$ is even and has an even DFT.
  Show that the following signals are even:
  $N = 4$, $x = \{a, b, c, b\}$, and
  $N = 5$, $x = \{a, b, c, c, b\}$.

- **4.** Odd functions $x(n)$. By definition:

$$x_{odd}(n) = -x(-n) = -x(N - n) \ . \tag{3.24}$$

  Prove that if $x(n)$ is odd so is $x(\omega_k)$:

$$x(\omega_k) = -x(-\omega_k) = -x(\omega_{-k}) = -x(\omega_{N-k}) \ . \tag{3.25}$$

Thus oddness is an intrinsic property of $|x>$ and is independent of the coordinate frame used. As an example show explicitly that $< n|x >= sin(\omega n)$ is odd and has an odd DFT. Show that $x_{odd}(0) = 0$. Show that if N is even then $x_{odd}(\frac{N_{even}}{2}) = 0$.
Show that the following signals are odd:
$N = 4$, $x = \{0, a, 0, -a\}$, and
$N = 5$, $x = \{0, a, b, -b, -a\}$.

- **5. (Circular/Cyclical/Periodic) CONVOLUTION:** We follow exactly the same steps as in Section(2.2).
  a) Consider the vectors $|f_1>$ and $|f_2>$. The vector $|F>$ is defined such that $F(n) = f_1(n)f_2(n)$. The question is to compute the DFT $< \omega_k|F >$. We proceed step by step:

$$
\begin{aligned}
< \omega_k|F > &= < \omega_k|1_{op}|F >= \sum_{n=0}^{N-1} < \omega_k|n >< n|F > \qquad (3.26)\\
&= \sum_{n=0}^{N-1} \frac{e^{-i\omega_k n}}{\sqrt{N}} < n|f_1 >< n|f_2 >\\
&= \sum_{n=0}^{N-1} \frac{e^{-i\omega_k n}}{\sqrt{N}} < n|1_{op}|f_1 >< n|1_{op}|f_2 >\\
&= \sum_{n=0}^{N-1} \frac{e^{-i\omega_k n}}{\sqrt{N}} \sum_{k_1} < n|\omega_{k_1} >< \omega_{k_1}|f_1 > \sum_{k_2} < n|\omega_{k_2} >< \omega_{k_2}|f_2 >
\end{aligned}
$$

we now change the order of summations

$$
= \sum_{k_1}\sum_{k_2} < \omega_{k_1}|f_1 >< \omega_{k_2}|f_2 > \frac{1}{N^{3/2}} \sum_n e^{in(\omega_{k_1}+\omega_{k_2}-\omega_k)}
$$

the sum over n is N times the Kronecker delta

$$
= \sum_{k_1}\sum_{k_2} < \omega_{k_1}|f_1 >< \omega_{k_2}|f_2 > \frac{1}{N^{3/2}} N\delta_{k_1+k_2-k}
$$

if we set $k_2 = k - k_1$ we get

$$
F(\omega_k) = \frac{1}{\sqrt{N}} \sum_{k_1=0}^{N-1} f_1(\omega_{k_1}) f_2(\omega_{k-k_1})
$$

on the other hand, if we set $k_1 = k - k_2$ we get

$$
F(\omega_k) = \frac{1}{\sqrt{N}} \sum_{k_2=0}^{N-1} f_1(\omega_{k-k_2}) f_2(\omega_{k_2}) \qquad (3.27)
$$

We now write this important result again:

$$
\begin{aligned}
\text{If } F(n) &= f_1(n) f_2(n) \text{ then,} &(3.28)\\
F(\omega_k) &= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_1(\omega_j) f_2(\omega_{k-j}) \\
&= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_1(\omega_{k-j}) f_2(\omega_j) &(3.29)
\end{aligned}
$$

and say, $F(\omega_k)$ is the convolution of $f_1(\omega_k)$ and $f_2(\omega_k)$, (where the convolution is defined by the sums in Eq.(3.29)).

b) Now the other way around: Consider the vectors $|f_1>$ and $|f_2>$. The vector $|F>$ is defined such that $F(\omega_k) = f_1(\omega_k) f_2(\omega_k)$. The question is to compute the Fourier transform $< n|F >$. Proceed step by step as in Eq.(3.26) and obtain the result:

$$
\begin{aligned}
\text{If } F(\omega_k) &= f_1(\omega_k) f_2(\omega_k) \text{ then,} &(3.30)\\
F(n) &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f_1(m) f_2(n-m) &(3.31)\\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f_1(n-m) f_2(m) \ ,
\end{aligned}
$$

and say, $F(n)$ is the convolution of $f_1(n)$ and $f_2(n)$.

- **6.** Real signal $x(n)$. In this case the DFTransform is

$$
\begin{aligned}
< n|x > &= \sum_k < n|\omega_k > x(\omega_k) \\
& \quad \text{take the complex conjugate} \\
< n|x > &= < n|x >^* \quad , \quad \text{(since } x(n) \text{ is real),} \\
&= \sum_k < n|\omega_{-k} > x^*(\omega_k) \\
& \quad \text{let the summation index } k \to -k \\
&= \sum_k < n|\omega_k > x^*(-\omega_k). &(3.32)
\end{aligned}
$$

Thus we obtain, if $x(n)$ is real then,

$$x(\omega_k) = x^*(-\omega_k),$$

and we can compare the real and the imaginary parts of the above equation

$$\begin{aligned} x_R(\omega_k) &= x_R(-\omega_k), \\ x_I(\omega_k) &= -x_I(-\omega_k). \end{aligned} \tag{3.33}$$

where x_R and x_I denote the real and the imaginary parts of $x(\omega_k)$. Thus the real part of the DFT of real $x(n)$ is symmetric and the imaginary part is antisymmetric functions. This makes their absolute magnitudes equal and their phase angles opposite in sign:

$$\begin{aligned} |x(\omega_k)| &= |x(\omega_{N-k})| \\ \angle\, x(\omega_k) &= -\angle\, x(\omega_{N-k}). \end{aligned} \tag{3.34}$$

Problem set: 5

Some properties of DFT:

1. If the sequence $\mathcal{MI}x$ is such that $\mathcal{MI}x(n) = x(-n)$, show that $\mathcal{MI}x(\omega_k) = x(\omega_{-k})$, that is, a mirror imaged time sequence has a mirror imaged DFT frequency sequence.

2. If the sequence x^* is such that $x^*(n) = [x(n)]^*$, show that $x^*(\omega_k) = ([\mathcal{MI}x](\omega_k))^*$, that is, DFT of a complex conjugate time-sequence, is complex conjugate of the DFT of the original sequence mirror-imaged.

3. If the sequence $\mathcal{MI}x^*$ is such that $\mathcal{MI}x^*(n) = (x(-n))^*$, show that $\mathcal{MI}x^*(\omega_k) = (x(\omega_k))^*$, that is, DFT of a mirror-imaged complex-conjugate time sequence, is simply the complex conjugate of the DFT of the original sequence.

4. If the sequence x_r is real, such that $x_r(n) = (x_r(n))^*$, show that it follows from **1, 2, 3** above that $x_r(\omega_k) = x_r^*(\omega_{-k}) = x_r^*(-\omega_k)$, that is, the magnitude of the DFT of a real signal is the same for mirror image pair of terms

$$|x_r(\omega_k)| = |x_r(-\omega_k)|\,. \tag{3.35}$$

When the time sequence is real, DFT magnitude $|x_r(\omega_k)|$ has mirror image symmetry. This is important in practice. Because we may discard all negative frequency spectral components and regenerate them later if needed from positive-frequency samples. Thus, the spectral plots of real signals are normally displayed only for positive frequencies over the range 0 Hz to $f_s/2$ Hz. On the other hand, the spectrum of a complex sequence must be shown from $-f_s/2$ to $f_s/2$ (or from 0 to f_s), since the positive and negative frequency components of a complex signal are independent.

- - - - - - - - - - - - -

(We change the normalization for the DFT's in problems 5 and 6 below.Let $x(n)$ denote an array with N elements in time domain. Then $\hat{X}(k)$ =DFT$[x](k)$ is given by

$$\begin{aligned}\hat{X}(k) &= \sum_{n=0}^{N-1} W^{kn}\, x(n) \\ x(n) &= \frac{1}{N}\sum_{k=0}^{N-1}(W^{nk})^* \hat{X}(k) \\ W^{kn} = W^{nk} &= e^{-i\frac{2\pi}{N}nk} \end{aligned} \tag{3.36}$$

Now problems 5 and 6:)

- - - - - - - - - - - - -

5. Upsampling Consider an array in time domain: $x = \{a, b, c, d\}$.
a) Compute the DFT $\hat{X}(k) = \sum_n W^{kn} x_n$, with $W = e^{-i\frac{2\pi}{4}}$. Show that

$$\begin{aligned} \text{if} \quad \hat{X} &= DFT(x) \quad \text{then} \\ \hat{X}(0) &= a + b + c + d \\ \hat{X}(1) &= a - ib - c + id \\ \hat{X}(2) &= a - b + c - d \\ \hat{X}(3) &= a + ib - c - id \end{aligned} \tag{3.37}$$

Note that if $x = \{a, b, c, d\}$ is real, then $\hat{X}(1) = \hat{X}^*(3)$, as required by (3.34 and 3.35).

b) An array y is defined as $y = \{a, 0, b, 0, c, 0, d, 0\}$. This is indicated by

$y = \mathcal{U}_2(x)$, where the operator $\mathcal{U}_2$ *upsamples* the signal x by two. Compute the DFT of the *upsampled* signal y. That is $\hat{Y} = \text{DFT}(y)$. Show that $\hat{Y} = \left(\hat{X}, \hat{X}\right)$.
c) Another array is $z = \{a, 0, 0, b, 0, 0, c, 0, 0, d, 0, 0\} = \mathcal{U}_3(x)$. Based on your result of part (b), write down the result you guess for $\hat{Z}$ (ans: $\hat{Z} = \left(\hat{X}, \hat{X}, \hat{X}\right)$.

6. Downsampling Consider an array in time domain: $x = \{a, b, c, d\}$. An array y is obtained from x as throwing away every second term: $y = \{a, c\}$. This is indicated by $y = \mathcal{D}_2(x)$, where the operator $\mathcal{D}_2$ *downsamples* the signal x by two. Compute the DFT(y)= $\hat{Y}$. Show that the DFTs are related in the following manner: define the aliasing operator $\mathcal{A}_L$,

$$\mathcal{A}_L \, \hat{X}(k) = \sum_{\ell=0}^{L-1} \hat{X}(k + \ell \frac{N}{L}). \tag{3.38}$$

Write $\hat{Y}$ in terms of aliased $\hat{X}$, (you will determine L, and N is the size of x).
(Answer: $N = 4$, $L = 2$,

$$\begin{aligned}
y &= \mathcal{D}_2(x) \\
\hat{Y} &= \text{DFT}\,(y) = \frac{1}{2} \, \mathcal{A}_2 \, \hat{X} \\
\hat{Y}(0) &= a + c = \frac{1}{2}\left(\hat{X}(0) + \hat{X}(2)\right) \\
\hat{Y}(1) &= a - c = \frac{1}{2}\left(\hat{X}(1) + \hat{X}(3)\right).
\end{aligned} \tag{3.39}$$

3.4 Spectral Leakage

Consider a sample $x(n) = e^{i\omega n}$ where the (dimensionless) angular frequency is *not* one of the DFT frequencies $\omega_k \equiv \frac{2\pi k}{N}$. The DFT of this signal is then given by,

$$\begin{aligned} < \omega_k|x > &= < \omega_k|1_{op}|x >= \sum_{n=0}^{N-1} < \omega_k|n >< n|x > \\ &= \sum_{n=0}^{N-1} \left(\frac{e^{-i\omega_k n}}{\sqrt{N}} \right) e^{iwn} \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(e^{i(\omega-\omega_k)} \right)^n \\ &= \frac{1}{\sqrt{N}} e^{\frac{i}{2}(\omega-\omega_k)(N-1)} \left\{ \frac{sin\frac{N}{2}(\omega-\omega_k)}{sin\frac{1}{2}(\omega-\omega_k)} \right\} \end{aligned} \tag{3.40}$$

where we summed the geometric series:

$$1 + a + a^2 + \ldots + a^{N-1} = \frac{1-a^N}{1-a}, \quad a = e^{i(\omega-\omega_k)}. \tag{3.41}$$

If $|x >= A|\omega_\ell >$, proportional to one of the basis sinusoids, then $< \omega_k|x >= A\delta_{k\ell}$. Namely the entire strength of the signal is on one basis sinusoid. If however the (dimensionless) angular frequency is *not* one of the DFT frequencies ω_k then the strength is distributed over many basis sinusoids, as indicated by Eq.(3.40). This is called "spectral leakage". Also note the "smearing function" $sinNz/sinz$ form appropriate for the finite N-dimensional case (rather than the $sinz/z$ form in the continuous case (see. Eq.(2.39)). See numerical Matlab examples on "spectral leakage".
Hence, our efforts of determining the frequency content of our signal is frustrated due to the spectral leakage. If we increase N, and increase the sampling frequency f_s, keeping the observation time $NT_s = T_o$ constant, still this is not a remedy. Because the basis sinusoids $|\omega_k >$ are such that exactly k periods fit in T_o. Since the frequency of a basis sinusoid is $f_k = kf_s/N$ it follows that

$$T_o = NT_s = kT_k \; . \tag{3.42}$$

But ω does not have integer number of cycles in T_o. Thus "glitch" is unavoidable as is shown in the MATLAB example "Spectral Leakage". This effect can be reduced by using a suitable *window* $W(n)$ which diminishes the data smoothly to zero at both endpoints of the window. We will consider window functions a bit later.

3.5 Handling finite length signals

Since DFT transforms and circular convolutions involve handling of finite length digital signals, we study the manipulations of such signals. Consider a signal $x = \{x_o, x_1, \ldots, x_{N-1}\}$. We associate *fixed* positions $m = 0, 1, \ldots, N-1$, on a circle with the N^{th} roots of unity. Also, initially place the components of the signal $x_m = x(m)$ at the positions of the roots $z_m = z_1^m$. Actions/transformations on the signal, will cause the occupants of the fixed addresses m on the circle to be signal components different than x_m. A general linear transformation will change the occupant of the address m : $x(m) \to x(a + \sigma m)$, where a is a positive or negative integer and σ is a sign $\pm$. We are interested in two types of operations

- **1. Circular permutations** A circular permutation is defined by:

$$\mathcal{C}\, x = \{x_1, x_2, \ldots, x_{N-1}, x_o\}. \tag{3.43}$$

If we define $x(.)$ to represent $x(n)$ for any n, then Eq.(3.43) can also be written as

$$\mathcal{C}x(.) = x(. + 1)\,. \tag{3.44}$$

If $x(n)$ is represented on the circle in the direction of z_n, a root of the unity, then $\mathcal{C}\; x(m) = x(m + 1)$, or the signal moved one notch in the clockwise direction on the circle. The occupant of the m^{th} position changes $x_m \to x_{m+1}$. The inverse operation is:

$$\begin{aligned} \mathcal{C}^{-1}\, x &= \{x_{N-1}, x_o, x_1, x_2, \ldots, x_{N-2}\}. \quad \text{or equivalently,} \\ \mathcal{C}^{-1}\, x(.) &= x(. - 1)\,. \end{aligned} \tag{3.45}$$

So that $\mathcal{C}^{-1}\; x(m) = x(m - 1)$, or the signal moved one notch in the anticlockwise direction on the circle. The occupant of the m^{th} position changes $x_m \to x_{m-1}$. We can combine the relations (3.43 and 3.45). Thus for λ a positive or negative integer, we obtain

$$\mathcal{C}^{\lambda}\; x(n) = x(n + \lambda). \tag{3.46}$$

- **2. Mirror imaging** This operation is defined by:

$$\mathcal{MI}\, x = \{x_o, x_{N-1}, x_{N-2}, \ldots, x_2, x_1\}. \tag{3.47}$$

Note that $\mathcal{MI}\, x(m) = x(-m)$. This means the signal components are reflected about the horizontal diameter of the circle, and $x_m \to x_{-m} = x_{N-m}$. Mirror imaging is its own inverse, $\mathcal{MI}\; \mathcal{MI} = 1$.

These two operations do not commute, and we observe that:

$$\mathcal{C}\,\mathcal{MI}\,x = \mathcal{MI}\,\mathcal{C}^{-1}\,x = \{x_{N-1}, x_{N-2}, \ldots, x_2, x_1, x_o\}. \tag{3.48}$$

Thus we obtain $\mathcal{C}\,\mathcal{MI}\,\mathcal{C} = \mathcal{MI}$. We can iterate this equation to obtain:

$$\mathcal{C}^\lambda\,\mathcal{MI}\,\mathcal{C}^\lambda = \mathcal{MI}\,. \tag{3.49}$$

Note the similarity between the relation (2.32) for the continuous case and the relation (3.49) for the digital case. Here the operation $\mathcal{C}^{-n}$ in the digital case corresponds to translation $\mathcal{T}_q$ in the continuous case. It is a good exercise to follow the action of these operators in Eq.(3.49) on a signal $x(n)$:

$$\begin{aligned}
\mathcal{C}^\lambda\,\mathcal{MI}\,\mathcal{C}^\lambda\,x(n) &= \mathcal{C}^\lambda\,\mathcal{MI}\,x(n+\lambda), \\
&= \mathcal{C}^\lambda\,x(-n+\lambda) \\
&= x(-(n+\lambda)+\lambda) = x(-n) \\
&= \mathcal{MI}\,x(n)\,.
\end{aligned} \tag{3.50}$$

We now write the action of our operators on a signal of the general form $x(a+\sigma n)$ where a is a positive or negative integer constant, n takes on the values $0, 1, \ldots, N-1$ and $\sigma = \pm 1$ represents the sign.

$$\begin{aligned}
\mathcal{MI}\,x(a+\sigma\,n) &= x(a-\sigma\,n), \\
\mathcal{C}^\lambda\,x(a+\sigma\,n) &= x(a+\sigma(n+\lambda)), \\
\mathcal{C}^{-\lambda}\,x(a+\sigma\,n) &= x(a+\sigma(n-\lambda))\,.
\end{aligned} \tag{3.51}$$

Problem set: 6

Let us now familiarize ourselves further with the shifted sequences used in convolutions and correlations with specific examples:
Let $x(m)$ be the ordered sequence $x(m) = \{1, 2, 3, 4, 5, 0, 0\}$ where there are $N = 7$ elements in the sequence and $x(0) = 1, x(1) = 2, \ldots, x(6) = 0$ and periodicity demands that $x(m+N) = x(m)$. Draw a circle and divide the

circumference by $N = 7$, and label inside , anticlockwise from 0 to 6 in the directions of the 7th roots of unity, and put the x values on the outside next to their respective positions.

1. Show that $x(m-n)$, (for n fixed, say $n = 2$, and $m = 0, 1, .., 6$), means the sequence is rotated by $n = 2$ notches in the counterclockwise direction. The pictures are easy on the circle but care is needed to show them as a linear sequence. For example:

$$x(m-2) = \{0012345\}.$$

Find a, b such that $\mathcal{C}^a \mathcal{M}\mathcal{I}^b x(m) = x(m-2)$ where $a = (1, \ldots, N-1)$ and $b = 0$ or 1.

2. Show that for $x(n-m)$ (for n fixed, say $n = 2$, and $m = 0, 1, .., 6$) the sequence is:

$$x(2-m) = \{3210054\}.$$

Find a, b such that $\mathcal{C}^a \mathcal{M}\mathcal{I}^b x(m) = x(2-m)$ where $a = (1, \ldots, N-1)$ and $b = 0$ or 1.

3. Show that for $x(m+n)$ (for n fixed, say $n = 2$, and $m = 0, 1, .., 6$) the sequence is:

$$x(m+2) = \{3450012\}.$$

Find a, b such that $\mathcal{C}^a \mathcal{M}\mathcal{I}^b x(m) = x(2+m)$ where $a = (1, \ldots, N-1)$ and $b = 0$ or 1.

4. Finally write a MATLAB program that computes all the sequences $x(2-m), x(m-2), x(2+m)$ for the signal $x(m) = \{1, 2, 3, 4, 5, 0, 0\}$. Then instead of 2 use an integer variable n for all seven possible values of n . Then compute $\mathcal{C}^{\lambda} x$ and $\mathcal{C}^{-\lambda} x$ for any λ, and $\mathcal{M}\mathcal{I}x$. You should also be able to compute $\mathcal{C}^{\lambda}\ \mathcal{M}\mathcal{I}\ \mathcal{C}^{\mu} x$. Also write a C-code for the same purpose.

Consider now the circular convolution formulas given by Eq.(3.31):

$$F(n) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f_1(m) f_2(n-m)\ .$$

We see that $f_2(n-m)$ is the *rotated* and *mirror-imaged* signal $f_2(m)$:

$$f_2(n-m) = \mathcal{C}^{-n}\,\mathcal{MI}\ f_2(m) = \mathcal{MI}\ \mathcal{C}^{n}\ f_2(m)\ . \tag{3.52}$$

Eq.(3.52) is the digital version of the continuous case formula Eq.(2.33) for the form of one of the pair of functions used in convolutions.

3.6 Zero Padding

Suppose we have data (sampled and windowed signal) $x(n)$ of length N, that is the index n runs from zero to $N-1$. If we add $M-N$ zero samples to the end of our data ($M > N$, this is called "zero padding". The zero padded data is given by the sequence $\tilde{x}(n)$:

$$\tilde{x}(n) = \begin{cases} x(n) & \text{for } n = 0, \ldots, N-1, \\ 0 & \text{for } n = N, \ldots, M-1. \end{cases} \tag{3.53}$$

Zero padding is needed in two places:

1. In order to increase the frequency resolution in spectral analysis.

   $$\Delta\omega_N = \frac{\omega_s}{N} \rightarrow \Delta\omega_M = \frac{\omega_s}{M}. \tag{3.54}$$

   where $\Delta\omega_N > \Delta\omega_M$.
   This allows us to determine the frequency of an isolated spectral peak to a desired accuracy $\Delta\omega_M$ by sufficient zero padding. However, to resolve two close peaks , our resolution is still $\Delta\omega_N$. This is forced on us by the fact that raw sampled data is being convolved with the window function in the frequency space, and that we cannot resolve the frequencies that fall under the main lobe of the window. So, zero padding is essentially an interpolation. See the Matlab examples.

2. Zero padding is again needed in order to avoid the wrap-around and overlap problems while using the circular convolution for the purpose of linear convolution as we shall study in the next section. In this case the two sequences of lengths N_1 and N_2 are zero padded to length $M = N_1 + N_2 - 1$. Again, see the Matlab examples.

3.7 Computation of Linear Convolutions using DFT-Circular Convolutions

Since DFT is a practical way of computing transforms, we should make use of it in the computation of linear convolutions which are needed in various applications. We learned to visualize shifted sequences in the exercises of Section 3.5 . We now want to compute the linear convolution of two signals: signal x_1 of length N_1 and signal x_2 of length N_2. We want to compute:

$$C(k) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(k-m) \tag{3.55}$$

Assume x_1 is fixed along the line, starting from origin and extending (and including) the site $N_1 - 1$. And consider the second signal approaching from the right and moving to the left in flipped form $x_2(k-m)$:

$$\mathcal{C}^{-k}\, \mathcal{MI}\, x_2(m) = \mathcal{MI}\, \mathcal{C}^{k}\, x_2(m) = x_2(k-m)\,. \tag{3.56}$$

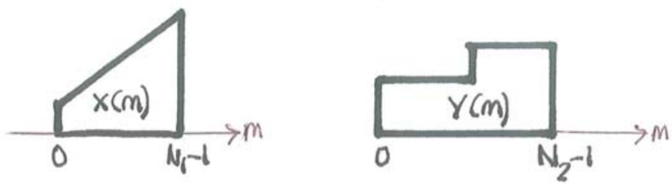
And we take snapshots (see Figure). Let k be a large integer, and denote the site occupied by the *far* end of the flipped form of the signal x_2. The near end of the flipped signal x_2 occupies the site $k - N_2 + 1$. If the near end of the flipped second signal is not overlapping with the far end of the first signal then $C(k) = 0$. The overlap of two signals is first possible when $k - N_2 + 1 = N_1 - 1$. Also as the flipped second signal moves left, $k = 0$ is the last position when there is a non-vanishing contribution to the convolution. And when $k = -1, -2$ etc. there is no overlap anymore. Hence we conclude that the maximum and the minimum k values that contribute to the convolution are:

$$\begin{aligned} k_{max} &= N_1 + N_2 - 2 \\ k_{min} &= 0. \end{aligned} \tag{3.57}$$

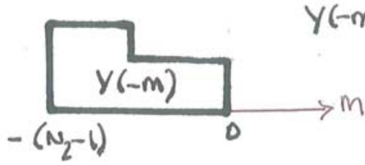
Thus the linear convolution is of length $N_1 + N_2 - 1$. Therfore in order to compute the convolution of x_1 and x_2, we must first zero pad them to length $N_1 + N_2 - 1$. Only then the circular convolution can be used reliably, and the problems of wrap around and overlap are avoided.

Linear Convolution

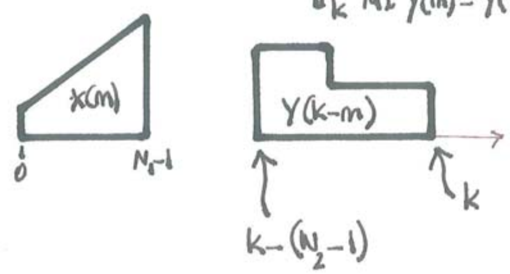
$$C(k) = \sum_{m=-\infty}^{+\infty} x(m)\, y(k-m)$$

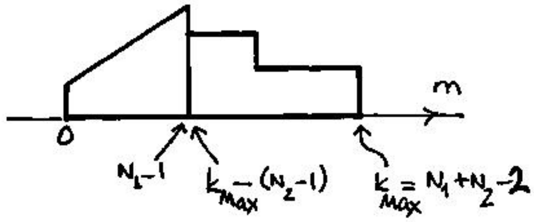


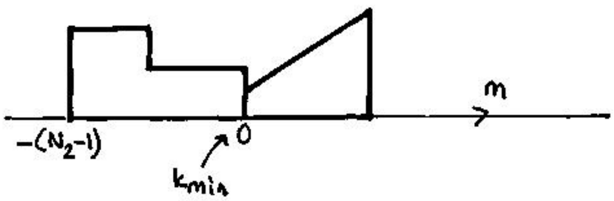
$Y(-m) = MI\ y(m)$



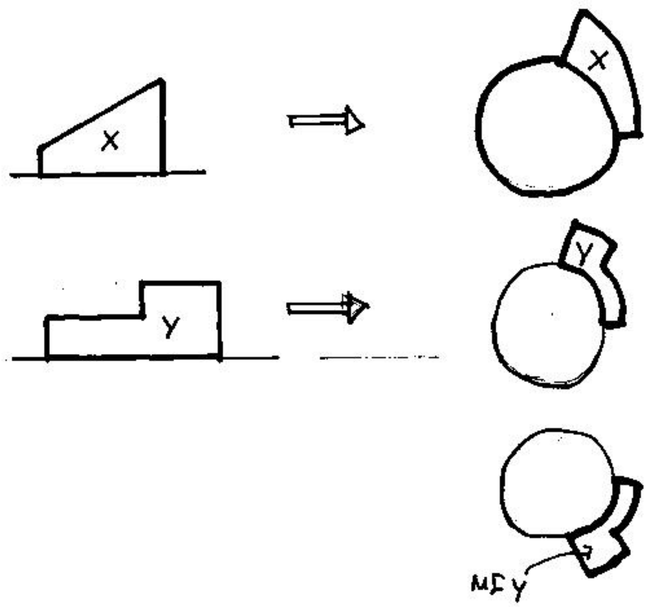
$T_k\ MI\ y(m) = y(k-m)$

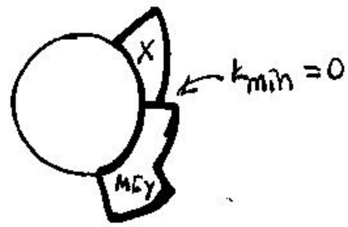


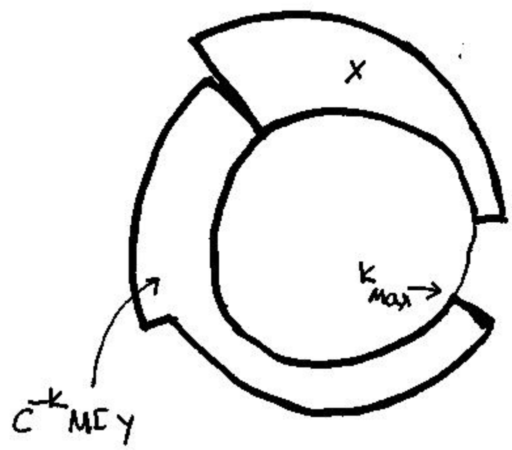


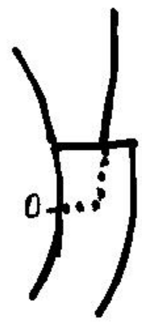


Circular Convolution









Avoid: overlap and wrap around.

$k_{max} = N_1 + N_2 - 2$

$k_{min} = 0$

To avoid wrap around and overlap

$k_{max} \leq N-1$

or $N_1 + N_2 - 1 \leq N$

Exercise: For $x = [12345]$ and $y = [2222333]$ compute the linear convolution C, and write a Matlab program for it. (You can of course compute the convolution directly by using the built-in function "conv"in Matlab, which should provide a check on your result.) Clearly both signals first have to be zero-padded to length $N_1 + N_2 - 1$ and only after then one should compute the linear convolution by employing circular convolution, in order to avoid the wrap around and overlap problem, (see the Matlab exercise "$q =$convolve(x, y)"). It is also obvious that linear convolution does not suffer from the "wrap around and overlap" problems of the circular convolution.

Problem set: 7

This problem set is based on the work of Julian Schwinger: [7]

1. In the case of continuous Fourier transforms (FT), we have seen that the Hermitian k_{op} generates infinitesimal translations of the $\{< q|\}$ states. For the discrete case, of course such an operator cannot be constructed. Try a $\mathcal{K}_{op}$ that does the following:

$$< q_j|\mathcal{K}_{op} =< q_{j+1}| \quad j = 0, \ldots, N-1. \tag{3.58}$$

a) Argue that $\mathcal{K}_{op}$ must be a unitary operator.

b) Use the periodic indexing of the $\{< q_j|\}$ states to prove that $\mathcal{K}_{op}$ satisfies the algebraic equation:

$$\mathcal{K}_{op}^N - 1_{op} = 0\,. \tag{3.59}$$

c) Argue that the eigenvalues $\{\mathcal{K}'_\ell\}$ of the $\mathcal{K}_{op}$ are the N^{th} roots of unity, $\mathcal{K}'_\ell = exp\left(i\frac{2\pi\ell}{N}\right)$ with $\ell = 0, 1, \ldots, N-1$ and show them as vectors in the complex plane.

d) Use the algebraic relation $a^N - 1 = (a-1)(1 + a + a^2 + \ldots + a^{N-1})$ for the operator relation Eq.(3.59), with $\mathcal{K}_{op}$ replaced by $\mathcal{K}_{op}/\mathcal{K}'$, where $\mathcal{K}'$ is an eigenvalue, to obtain

$$(\mathcal{K}_{op} - \mathcal{K}')\,\frac{1}{N}\sum_{j=0}^{N-1}\left(\frac{\mathcal{K}_{op}}{\mathcal{K}'}\right)^j = 0\,. \tag{3.60}$$

We now define operator valued Kronecker delta :

$$\delta_{op}(\mathcal{K}_{op},\,\mathcal{K}') = \frac{1}{N}\sum_{j=0}^{N-1}\left(\frac{\mathcal{K}_{op}}{\mathcal{K}'}\right)^j\,, \tag{3.61}$$

The merit of this operator is that $\delta_{op}(\mathcal{K}_{op},\,\mathcal{K}') \to 1_{op}$ as $\mathcal{K}_{op} \to \mathcal{K}'1_{op}$.

e) Show that the operator $\delta_{op}(\mathcal{K}_{op},\,\mathcal{K}')$ is Hermitian.

f) Construct the $< \mathcal{K}'|$ states as:

$$< \mathcal{K}'_\ell| = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j\ell} < q_j| \quad \ell = 0, \dots, N-1. \tag{3.62}$$

Show that this state is an eigenvector of the $\mathcal{K}_{op}$,

$$< \mathcal{K}'_\ell|\mathcal{K}_{op} = \mathcal{K}'_\ell < \mathcal{K}'_\ell| = e^{i\frac{2\pi\ell}{N}} < \mathcal{K}'_\ell| \,. \tag{3.63}$$

Show that if Eq(3.63) is true, then the following equation also holds true,

$$\mathcal{K}_{op}|\mathcal{K}' >= \mathcal{K}'|\mathcal{K}' > \,, \tag{3.64}$$

hence the states $|\mathcal{K}' >$ and their hermitian conjugates are the right and left eigenvectors of the operator $\mathcal{K}_{op}$ with the (complex) eigenvalue $\mathcal{K}'$.
Thus the $< \mathcal{K}'|$ states are the eigenvectors of a unitary operator stepping up through $\{< q_j|\}$ states.

g) Show that the $\{< \mathcal{K}'|\}$ states are orthonormal

$$< \mathcal{K}'|\mathcal{K}'' >= \delta(\mathcal{K}', \mathcal{K}'') \,, \tag{3.65}$$

and the unity operator of the vector space can be written in terms of them also:

$$1_{op} = \sum_{\ell=0}^{N-1} |\mathcal{K}'_\ell >< \mathcal{K}'_\ell| \,. \tag{3.66}$$

h) Show that

$$< \mathcal{K}''|\delta_{op}(\mathcal{K}_{op}, \mathcal{K}') = \delta(\mathcal{K}', \mathcal{K}'') < \mathcal{K}'| \,. \tag{3.67}$$

Hence show that the Kronecker delta operator has the attributes of a projection operator:

$$\delta_{op}(\mathcal{K}_{op}, \mathcal{K}') \; \delta_{op}(\mathcal{K}_{op}, \mathcal{K}'') = \delta(\mathcal{K}', \mathcal{K}'') \; \delta_{op}(\mathcal{K}_{op}, \mathcal{K}') \,. \tag{3.68}$$

Thus the Kronecker delta operator can be written as an outer product:

$$\delta_{op}(\mathcal{K}_{op}, \mathcal{K}') = |\mathcal{K}' >< \mathcal{K}'| \,. \tag{3.69}$$

2. In a similar way, we introduce the operator $\mathcal{Q}$ that steps up through the $|\mathcal{K}' >$ states:

$$\mathcal{Q}_{op}|\mathcal{K}'_\ell >= |\mathcal{K}'_{\ell+1} > , \quad \ell = 0, \dots, N-1, \tag{3.70}$$

where we chose it to step through the "kets", (compare with Eq.(**??**)), rather than the "bras", guided by the forms of $< q|k_{op}$ and $q_{op}|k >$ in the continuous case.

a) Argue that $\mathcal{Q}_{op}$ must be a unitary operator.

b) Show that the states $\{|\mathcal{K}'_\ell >\}$ are periodic indexed modulo N and prove that $\mathcal{Q}_{op}$ satisfies the algebraic equation:

$$\mathcal{Q}_{op}^N - 1_{op} = 0\,. \tag{3.71}$$

c) Argue that the eigenvalues $\{\mathcal{Q}'_j\}$ of the $\mathcal{Q}_{op}$ are the N^{th} roots of unity, and write them down explicitly.

d) Use the algebraic relation $a^N - 1 = (a-1)(1 + a + a^2 + \ldots + a^{N-1})$ for the operator relation Eq.(3.71), with $\mathcal{Q}_{op}$ replaced by $\mathcal{Q}_{op}/\mathcal{Q}'$, where $\mathcal{Q}'$ is an eigenvalue, to obtain

$$(\mathcal{Q}_{op} - \mathcal{Q}')\,\frac{1}{N}\sum_{j=0}^{N-1}\left(\frac{\mathcal{Q}_{op}}{\mathcal{Q}'}\right)^j = 0\,. \tag{3.72}$$

We now define operator valued Kronecker delta :

$$\delta_{op}(\mathcal{Q}_{op},\,\mathcal{Q}') = \frac{1}{N}\sum_{j=0}^{N-1}\left(\frac{\mathcal{Q}_{op}}{\mathcal{Q}'}\right)^j\,, \tag{3.73}$$

The merit of this operator is that $\delta_{op}(\mathcal{Q}_{op},\,\mathcal{Q}') \to 1_{op}$ as $\mathcal{Q}_{op} \to \mathcal{Q}' 1_{op}$.

e) Show that the operator $\delta_{op}(\mathcal{Q}_{op},\,\mathcal{Q}')$ is Hermitian.

f) Construct the $|\mathcal{Q}' >$ states as:

$$|\mathcal{Q}'_j >= \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}j\ell}|\mathcal{K}'_\ell > \qquad \ell = 0, \ldots, N-1. \tag{3.74}$$

Show that this state is an eigenvector of the $\mathcal{Q}_{op}$,

$$\mathcal{Q}_{op}|\mathcal{Q}'_j >= \mathcal{Q}'_j|\mathcal{Q}'_j >= e^{i\frac{2\pi j}{N}}|\mathcal{Q}'_j > \ . \tag{3.75}$$

Show that if Eq(3.75) is true, then the following equation also holds true,

$$< \mathcal{Q}'|\mathcal{Q}_{op} = \mathcal{Q}' < \mathcal{Q}'|\,, \tag{3.76}$$

hence the states $|\mathcal{Q}'>$ and their hermitian conjugates are the right and left eigenvectors of the operator $\mathcal{Q}_{op}$ with the (complex) eigenvalue $\mathcal{Q}'$.
Thus the $|\mathcal{Q}'>$ states are the eigenvectors of a unitary operator stepping up through $\{|\mathcal{K}'_\ell>\}$ states.

g) Show that the $\{|\mathcal{Q}'>\}$ states are orthonormal

$$< \mathcal{Q}'|\mathcal{Q}''>= \delta(\mathcal{Q}', \mathcal{Q}'')\,, \tag{3.77}$$

and the unity operator of the vector space can be written in terms of them also:

$$1_{op} = \sum_{j=0}^{N-1} |\mathcal{Q}'_j>< \mathcal{Q}'_j|\,. \tag{3.78}$$

h) Show that

$$\delta_{op}(\mathcal{Q}_{op},\, \mathcal{Q}')|\mathcal{Q}''>= \delta(\mathcal{Q}', \mathcal{Q}'')|\mathcal{Q}'> \ . \tag{3.79}$$

Hence show that the Kronecker delta operator has the attributes of a projection operator:

$$\delta_{op}(\mathcal{Q}_{op},\, \mathcal{Q}')\; \delta_{op}(\mathcal{Q}_{op},\, \mathcal{Q}'') = \delta(\mathcal{Q}', \mathcal{Q}'')\; \delta_{op}(\mathcal{Q}_{op},\, \mathcal{Q}')\,. \tag{3.80}$$

Thus the Kronecker delta operator can be written as an outer product:

$$\delta_{op}(\mathcal{Q}_{op},\, \mathcal{Q}') = |\mathcal{Q}'>< \mathcal{Q}'|\ . \tag{3.81}$$

3. Insert the expression for $|\mathcal{K}'_\ell>$ given in Eq.(3.62), into the expression for $|\mathcal{Q}'_j>$ given by Eq.(3.74), use the sum rule for the N^{th} roots of unity, and obtain:

$$|\mathcal{Q}'_j>= |q_j> \ . \tag{3.82}$$

We now summarize the key equations:

$$\begin{aligned}
<q_j|\mathcal{K}_{op} &= <q_{j+1}|\,, \quad j=0,\ldots,N-1. \\
\mathcal{K}_{op}^N &= 1_{op} \\
\mathcal{K}_{op}|\mathcal{K}'> &= \mathcal{K}'|\mathcal{K}'> \\
<\mathcal{K}'_\ell| &= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j\ell} <q_j|\,, \quad \ell=0,\ldots,N-1. \\
\mathcal{Q}_{op}|\mathcal{K}'_\ell> &= |\mathcal{K}'_{\ell+1}>\,, \quad \ell=0,\ldots,N-1, \\
\mathcal{Q}_{op}^N &= 1_{op} \\
\mathcal{Q}_{op}|q_j> &= \mathcal{Q}'_j|q_j>\,,
\end{aligned}$$

hence we can relabel the state $|q_j>$ as $|\mathcal{Q}'_j>$ (however note that, $\mathcal{Q}'_j \neq q_j$ but is one of the N^{th} roots of unity)

$$\begin{aligned}
|q_j> &= |\mathcal{Q}'_j> \\
|\mathcal{Q}'_j> &= \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}j\ell}\,|\mathcal{K}'_\ell> \qquad \ell=0,\ldots,N-1. \qquad (3.83)
\end{aligned}$$

We see that the bases $\{|\mathcal{Q}'_j>\}$ and $\{|\mathcal{K}'_\ell>\}$ are just the DFT (Discrete Fourier Transform) bases.

a) Show that the overlap is:

$$<\mathcal{Q}'_j|\mathcal{K}'_\ell>= \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}j\ell}\,. \qquad (3.84)$$

b) Compare the two operations

- $\mathcal{Q}_{op}\,\mathcal{K}_{op}|\mathcal{K}'_\ell>$, and

- $\mathcal{K}_{op}\, \mathcal{Q}_{op}|\mathcal{K}'_\ell > .$

Show that the comparison allows us to write the operator relation:

$$\mathcal{K}_{op}\, \mathcal{Q}_{op} = e^{i\frac{2\pi}{N}}\, \mathcal{Q}_{op}\, \mathcal{K}_{op}\, . \tag{3.85}$$

This is interesting! The operators $\mathcal{Q}_{op}$ and $\mathcal{K}_{op}$ do not commute. Only in the limit $N \to \infty$ that they commute (classical limit). In the other extreme, $N = 2$ these operators anticommute, $\mathcal{K}_{op}\, \mathcal{Q}_{op} = -\mathcal{Q}_{op}\, \mathcal{K}_{op}$.

c) We can easily extend the computation in part 2C-b above. Compare the two operations

- $\mathcal{Q}^n_{op}\, \mathcal{K}^m_{op}|\mathcal{K}'_\ell >$, and
- $\mathcal{K}^m_{op}\, \mathcal{Q}^n_{op}|\mathcal{K}'_\ell > .$

Show that the comparison allows us to write the operator relation:

$$\mathcal{K}^m_{op}\, \mathcal{Q}^n_{op} = e^{i\frac{2\pi}{N}mn}\, \mathcal{Q}^n_{op}\, \mathcal{K}^m_{op}\, . \tag{3.86}$$

d) Consider an arbitrary operator F_{op} in $\mathcal{C}^N$. Show that it can be written as

$$\begin{aligned} F_{op} &= 1_{op}\, F_{op}\, 1_{op} = \sum_{a,b} |\mathcal{K}'_a > F_{ab} < \mathcal{K}'_b|\, , \\ \text{where} \quad F_{ab} &= < \mathcal{K}'_a|F_{op}|\mathcal{K}'_b > \; . \end{aligned} \tag{3.87}$$

Show that the operator $|\mathcal{K}'_a >< \mathcal{K}'_b|$ in the sum can be written as:

$$|\mathcal{K}'_a >< \mathcal{K}'_b| = \mathcal{Q}^{a-b}_{op} \delta_{op}(\mathcal{K}_{op},\, \mathcal{K}'_b) = \frac{1}{N} \sum_{j=0}^{N-1} \frac{\mathcal{Q}^{a-b}_{op}\, \mathcal{K}^j_{op}}{\mathcal{K}'^j_b}\, , \tag{3.88}$$

where we used the relation (3.69). Now insert this into Eq.(3.87) and show that the operator F_{op} can be written in terms of $\mathcal{Q}_{op}$ and $\mathcal{K}_{op}$:

$$F_{op} = f(\mathcal{Q}_{op}, \mathcal{K}_{op}) = \sum_{m,n} f_{mn}\, \mathcal{Q}^m_{op}\, \mathcal{K}^n_{op}\, . \tag{3.89}$$

e) Consider the *trace* of an operator F_{op} expressed as in (3.89),

$$
\begin{aligned}
Tr(F_{op}) = Tr\left\{f(\mathcal{Q}_{op}, \mathcal{K}_{op})\right\} &= \sum_{\mathcal{Q}'} < \mathcal{Q}'|F_{op}|\mathcal{Q}' > \\
&= \sum_{\mathcal{Q}'} < \mathcal{Q}'|F_{op}1_{op}|\mathcal{Q}' > \\
&= \sum_{\mathcal{Q}'\,\mathcal{K}'} < \mathcal{Q}'|F_{op}|\mathcal{K}' >< \mathcal{K}'|\mathcal{Q}' > \\
&= \sum_{\mathcal{Q}'\,\mathcal{K}'} < \mathcal{Q}'|\sum_{m,n} f_{mn}\, \mathcal{Q}_{op}^{m}\, \mathcal{K}_{op}^{n}|\mathcal{K}' >< \mathcal{K}'|\mathcal{Q}' > \\
&= \sum_{\mathcal{Q}'\,\mathcal{K}'} \sum_{m,n} f_{mn}\, \mathcal{Q}_{op}'^{m}\, \mathcal{K}'^{n} < \mathcal{Q}'|\mathcal{K}' >< \mathcal{K}'|\mathcal{Q}' > \\
&= \sum_{\mathcal{Q}'\,\mathcal{K}'} \sum_{m,n} f_{mn}\, \mathcal{Q}_{op}'^{m}\, \mathcal{K}'^{n} | < \mathcal{K}'|\mathcal{Q}' > |^2 \\
&= \frac{1}{N} \sum_{\mathcal{Q}'\,\mathcal{K}'} \sum_{m,n} f_{mn}\, \mathcal{Q}_{op}'^{m}\, \mathcal{K}'^{n} \\
&= \frac{1}{N} \sum_{\mathcal{Q}'\,\mathcal{K}'} f(\mathcal{Q}', \mathcal{K}')\,, \qquad (3.90)
\end{aligned}
$$

where we used the overlap relation Eq.(3.84). Now compute the trace of the operator

$$
Tr(\mathcal{Q}_{op}^{m}\, \mathcal{K}_{op}^{n}) = N\delta_{m,0}\ \delta_{n,0}\ , \qquad (3.91)
$$

to obtain the result on the right hand side make use of the sum rule obeyed by the N^{th} roots of unity.

Make use of the relation (3.91), and show that the orthonormality statement of the N^2 operators regarded as vectors in the operator space is,

$$
\frac{1}{N} Tr\left\{\left(\mathcal{Q}_{op}^{m}\, \mathcal{K}_{op}^{n}\right)^{\dagger} \left(\mathcal{Q}_{op}^{m'}\, \mathcal{K}_{op}^{n'}\right)\right\} = \delta_{mm'}\delta_{nn'}\ . \qquad (3.92)
$$

Thus $\mathcal{Q}_{op}$ and $\mathcal{K}_{op}$ are the generators of the complete orthonormal operator basis, the N^2 operators given below

$$
\frac{\mathcal{Q}_{op}^{m}\, \mathcal{K}_{op}^{n}}{\sqrt{N}}\ , \quad m, n = 0, \ldots, N-1\ . \qquad (3.93)
$$

To show the completeness of the basis operator set, consider

$$\hat{x} = \frac{1}{N}\sum_{m,n}(\mathcal{Q}_{op}^m\,\mathcal{K}_{op}^n)^\dagger X_{op}(\mathcal{Q}_{op}^m\,\mathcal{K}_{op}^n)\,. \tag{3.94}$$

You will show that

$$\frac{1}{N}\sum_{m,n}(\mathcal{Q}_{op}^m\,\mathcal{K}_{op}^n)^\dagger X_{op}(\mathcal{Q}_{op}^m\,\mathcal{K}_{op}^n) = 1_{op}Tr(X_{op})\,. \tag{3.95}$$

In order to obtain the result above, consider the matrix element $<\mathcal{K}_a'|\hat{x}|\mathcal{K}_b'>$.
I will indicate the steps and you will fill in the intermediate steps:

$$<\mathcal{K}_a'|\hat{x}|\mathcal{K}_b'> \;=\; \frac{1}{N}\sum_{m,n}\mathcal{K}_a'^{-n} <\mathcal{K}_a'|\mathcal{Q}_{op}^{-m}X_{op}\mathcal{Q}_{op}^m|\mathcal{K}_b'> \mathcal{K}_b'^n$$

do the sum over n (use the sum rule obeyed by roots of unity)

insert 1_{op} twice

$$= \delta_{ab}\sum_m\sum_{cd} <\mathcal{K}_a'|\mathcal{Q}_c'><\mathcal{Q}_c'|\mathcal{Q}_{op}^{-m}X_{op}\mathcal{Q}_{op}^m|\mathcal{Q}_d'><\mathcal{Q}_d'|\mathcal{K}_a'>$$

$$= \delta_{ab}\sum_m\sum_{cd} <\mathcal{K}_a'|\mathcal{Q}_c'>\mathcal{Q}_c'^{-m}<\mathcal{Q}_c'|X_{op}|\mathcal{Q}_d'>\mathcal{Q}_d'^m<\mathcal{Q}_d'|\mathcal{K}_a'>$$

do the sum over m (use the sum rule obeyed by roots of unity)

$$= \delta_{ab}N\sum_c|<\mathcal{K}_a'|\mathcal{Q}_c'>|^2<\mathcal{Q}_c'|X_{op}|\mathcal{Q}_c'>$$

observe that the overlap $<\mathcal{K}_a'|\mathcal{Q}_c'>$ is given in Eq.(3.84)

$$= \delta_{ab}Tr(X_{op})$$

$$<\mathcal{K}_a'|\hat{x}|\mathcal{K}_b'> \;=\; <\mathcal{K}_a'|1_{op}|\mathcal{K}_b'>Tr(X_{op}). \tag{3.96}$$

Since Eq.(3.96) holds for arbitarary matrix elements, we can write it as given above by Eq.(3.95). It is a very good exercise to redo the computation starting from the matrix element $<\mathcal{Q}_a'|\hat{x}|\mathcal{Q}_b'>$. Do it!

4. The unitary operator W is defined by

$$< Q'_j|W =< K'_j| \ , \quad j = 0, \ldots, N-1 \ . \tag{3.97}$$

a) Show, using the definition above, that

$$< K'_j|W =< Q'_{-j}| =< Q'_{N-j}| \ . \tag{3.98}$$

b) Start from the defining equation of $\mathcal{Q}_{op}$, : $< \mathcal{K}'_a|\mathcal{Q}_{op}^{-j} =< \mathcal{K}'_{a+j}|$. Act by W on these bras. Use Eq.(3.98), to obtain

$$< \mathcal{K}'_a|\mathcal{Q}_{op}^{-j}W =< \mathcal{K}'_a|W\mathcal{K}_{op}^{-j} \ . \tag{3.99}$$

Thus conclude that

$$W^{-1}\mathcal{Q}_{op}W = \mathcal{K}_{op} \ . \tag{3.100}$$

c) Start from the defining equation of $\mathcal{K}_{op}$, : $< \mathcal{Q}'_a|\mathcal{K}_{op}^{j} =< \mathcal{Q}'_{a+j}|$. Act by W on these bras. Use Eq.(3.97), to obtain

$$< \mathcal{Q}'_a|\mathcal{K}_{op}^{j}W =< \mathcal{Q}'_a|W\mathcal{Q}_{op}^{-j} \ . \tag{3.101}$$

Thus conclude that

$$W^{-1}\mathcal{K}_{op}W = \mathcal{Q}_{op}^{-1} \ . \tag{3.102}$$

d) Finally note that relations (3.100,3.102,3.85,3.86) are all *invariant* onder the interchange: $\mathcal{Q}_{op} \to \mathcal{K}_{op}$ and $\mathcal{K}_{op} \to \mathcal{Q}_{op}^{-1}$.

3.8 Spectral analysis & Periodogram

A simple approach first:
The purpose of spectral analysis is to study the distribution of power contained in the signal over the frequencies. An audio signal, in addition to deterministic components (known sine waves for example) may in general contain random (noise) components as well. One of the techniques used in this case is the method of periodogram.

But first let us consider *resolution*, our ability to resolve two neighboring frequencies: Clearly, the difference between the two frequencies $\Delta f = f_1 - f_2$ must be greater than the width of the mainlobe of the leaked spectra for either one of these sinusoids. Let us estimate the width of the mainlobe: The continuous signal is sampled in time at every time interval of length T_s for a duration of T_o . Hence the sample contains $N = T_o/T_s$ elements. Or, the duration of observing the signal is $T_o = NT_s$. The time frequency uncertainty product gives $\Delta t \Delta f > 1$. Here we identify Δt with T_o. Thus we obtain,

$$\Delta f > \frac{1}{T_o} = \frac{1}{NT_s} = \frac{f_s}{N}. \tag{3.103}$$

Meaning that we cannot resolve frequencies whose difference is less than $\frac{f_s}{N}$. Hence we express this estimate as:

$$f_1 - f_2 > \frac{f_s}{N}. \tag{3.104}$$

Example: Suppose the sampling rate is $f_s = 1000Hz$. If we want to be able to resolve a frequency difference of say, 10 Hz, then $N > 1000/10 = 100$, in words: our sampled signal must have a length greater than 100 samples, sampling rate being kept the same. In other words, we must observe this signal for more than 0.1 sec in order to be able to resolve a 10 Hz. difference.

Periodogram Let $x(n)$ denote a sampled and windowed signal of length N where $x(n) = x_a(n)W(n)$ with $x_a(n)$ and $W(n)$ denoting the sampled analog signal and the window function respectively. The discrete -time Fourier transform is :

$$< \omega|x >= \sum_{n=0}^{N-1} \left(\frac{e^{-in\omega}}{\sqrt{N}} \right) x_a(n)W(n). \tag{3.105}$$

The power density spectrum is then,

$$\begin{aligned} I(w) &\propto |<\omega|x>|^2 \\ &= \frac{1}{N}\sum_{n,m} x_a(n)^* W(n) x_a(m) W(m) e^{-i(m-n)\omega}. \end{aligned} \tag{3.106}$$

Notice that the difference $k \equiv m - n$ in the exponent goes from $-(N-1)$ to $N-1$. Hence the following sum, for fixed values of n and m, equals unity:

$$\sum_{k=-(N-1)}^{N-1} \delta(k, m-n) = 1. \tag{3.107}$$

Now multiply the right-hand side of Eq.(3.106) with unity as written above, and change the order of summations, doing the m sum first. The result is:

$$I(\omega) \propto \sum_{k=-(N-1)}^{N-1} e^{-ik\omega} C(k), \tag{3.108}$$

where $C(k)$ is an aperiodic autocorrelation function with $2N-1$ elements:

$$C(k) \equiv \sum_{n=0}^{N-1} x_a(n)^* W(n) x_a(n+k) W(n+k). \tag{3.109}$$

Thus, a periodogram is basically the Fourier transform of the autocorrelation function. It is clear that the periodogram is to be evaluated at the DFT frequencies $\omega_j = 2\pi j/L$, where L is the dimension of the FFT space (size of the Fast Fourier Transform). If L is larger than the sampled and windowed signal size N, then the signal has to be zero-padded.

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